**Example: Ordinary Point**

1. Solve $y'' - xy' + 2y = 0$ about the ordinary point $x = 0$ (we also did this in class)

   Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation,

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

   Rewrite so that every sum begins with the same power of $x$. In this case, we start with $x^1$ because of the middle sum. Rewriting the first two sums gives us:

$$\left(2c_2 + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k\right) - \sum_{k=1}^{\infty} kc_k x^k + \left(2c_0 + 2\sum_{k=1}^{\infty} c_k x^k\right) = 0$$

   And simplify:

$$2(c_2 + c_0) + \sum_{k=1}^{\infty}\left((k+2)(k+1)c_{k+2} - (k-2)c_k\right)x^k = 0$$

   This gives us the recurrence relation for the coefficients:

$$c_2 = -c_0 \quad \text{and} \quad c_{k+2} = \frac{k-2}{(k+2)(k+1)}c_k, \quad k = 1, 2, 3, \ldots$$

   With $c_0 = 1$, and $c_1 = 0$, we should find that $c_2 = -1$ and the rest of the coefficients are zero. Therefore, the first solution is:

$$y_1(x) = 1 - x^2$$

   With $c_0 = 0$ and $c_1 = 1$, we should find that

$$c_2 = 0,\ c_3 = -\frac{1}{6},\ c_4 = 0,\ c_5 = -\frac{1}{120}, \ldots$$

   Now,

$$y_2(x) = x - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \ldots$$

   and the general solution is $y = c_0 y_1 + c_1 y_2$.

2. (Problem 18, Section 6.1 for Extra Practice) Solve $y'' - xy = 0$ about $x = 0$.

   Going through our substitution, we get:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

   Rewrite so that the sums both start with $x^1$ (because of the second sum):

$$2c_2 + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} c_{k-1}x^k = 0$$

   And simplify:

$$2c_2 + \sum_{k=1}^{\infty}\left((k+2)(k+1)c_{k+2} - c_{k-1}\right)x^k = 0$$

Therefore, $c_2 = 0$ and

$$c_{k+2} = \frac{1}{(k+2)(k+1)} c_{k-1}, \ k = 1, 2, 3, \ldots$$

To get our linearly independent solutions, first let $c_0 = 1$, $c_1 = 0$ to get:

$$c_3 = \frac{1}{6}, \ c_4 = 0, \ c_5 = 0, \ c_6 = \frac{1}{180}$$

And now let $c_0 = 0$, $c_1 = 1$ to get the second solution:

$$c_3 = 0, \ c_4 = \frac{1}{12}, \ c_5 = c_6 = 0, \ c_7 = \frac{1}{504}$$

The two solutions are:

$$y_1 = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + h.o.t, \quad y_2 = x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + h.o.t$$

**Examples: Regular Singular Point**

3. Solve $y'' - \frac{1}{2x}y' + \frac{1+x}{2x^2}y = 0$ using the power series method about $x = 0$.

First, we see that:

$$P(x) = -\frac{1}{2x}, \quad Q(x) = \frac{1+x}{2x^2}$$

and we verify that $x = 0$ is a regular singular point. We do this by checking that $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$:

$$xP(x) = -\frac{1}{2}, \quad x^2Q(x) = \frac{1}{2}(1+x)$$

Both of these are analytic at $x = 0$, so $x = 0$ is a regular singular point.

Our technique will be to first multiply the equation by $x^2$, and we'll multiply by 2 to simplify the algebra:

$$2x^2y'' - xy' + (1+x)y = 0$$

Now assume the form of the solution to be the following. We've also include the first few terms for later:

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{r+n} = c_0x^r + c_1x^{r+1} + c_2x^{r+2} + \ldots$$

Previously, when we differentiated, we re-started the index (because the constant terms drop out), but now the constant terms will not drop out:

$$y' = \sum_{n=0}^{\infty} c_n(r+n)x^{r+n-1}, \quad y'' = \sum_{n=0}^{\infty} c_n(r+n)(r+n-1)x^{r+n-2}$$

Substitute into the differential equation:

$$2x^2 \sum_{n=0}^{\infty} c_n(r+n)(r+n-1)x^{r+n-2} - x\sum_{n=0}^{\infty} c_n(r+n)x^{r+n-1} + (1+x)\sum_{n=0}^{\infty} c_nx^{r+n} = 0$$

Simplifying:

$$\sum_{n=0}^{\infty} c_n(r+n)(r+n-1)x^{r+n} - \sum_{n=0}^{\infty} c_n(r+n)x^{r+n} + \sum_{n=0}^{\infty} c_nx^{r+n} + \sum_{n=0}^{\infty} c_nx^{r+n+1} = 0$$

The last sum could be rewritten (let $k = n + 1$, then let $n = k$) as

$$\sum_{n=1}^{\infty} c_{n-1} x^{r+n}$$

The first three sums begin with $x^r$, the last one starts with $x^{r+1}$. We split off the first entry from the first three sums, and combine the remaining sums to get:

$$(2c_0 r(r-1) - c_0 r + c_0) + \sum_{n=1}^{\infty} \left(2c_n(r+n)(r+n-1) - (r+n)c_n + c_n + c_{n-1}\right) x^{r+n} = 0$$

The constant term tells us what $r$ should be (THE INDICIAL EQUATION):

$$c_0(2r(r-1) - r + 1) = 0 \Rightarrow 2r^2 - 2r - r + 1 = 0 \Rightarrow 2r^2 - 3r + 1 = 0 \Rightarrow r = 1, \frac{1}{2}$$

We will get two linearly independent solutions by using the two values of $r$. First, for $r = 1$, we are back to our usual situation:

$$\sum_{n=1}^{\infty} (2c_n(n+1)n - nc_n + c_{n-1}) x^{n+1} = 0$$

which simplifies:

$$\sum_{n=1}^{\infty} (nc_n(2n+1) + c_{n-1}) x^{n+1} = 0$$

which implies that each coefficient must be zero:

$$c_n = -\frac{c_{n-1}}{n(2n+1)}$$

Notice that, rather than depending on both $c_0$ and $c_1$, we will now only depend on one of them. Let $c_0 = 1$, and compute the remaining coefficients, written out so you can see the pattern:

$$c_1 = -\frac{1}{3 \cdot 1}, \quad c_2 = -\frac{c_1}{2 \cdot 5} = \frac{1}{(3 \cdot 5) \cdot (1 \cdot 2)}, \quad c_3 = -\frac{c_2}{3 \cdot 7} = -\frac{1}{(3 \cdot 5 \cdot 7) \cdot (1 \cdot 2 \cdot 3)}$$

For the actual series, simplify to:

$$c_0 = 1, c_1 = -\frac{1}{3}, c_2 = \frac{1}{30}, c_3 = -\frac{1}{630}, \ldots$$

Similarly, solve when $r = \frac{1}{2}$. In this case, we have:

$$\sqrt{x} \sum_{n=1}^{\infty} \left( 2c_n \left(\frac{1}{2} + n\right) \left(n - \frac{1}{2}\right) - \left(n + \frac{1}{2}\right) c_n + c_n + c_{n-1} \right) x^n = 0$$

which simplifies to:

$$\sqrt{x} \sum_{n=1}^{\infty} \left(c_n (2n-1) n + c_{n-1}\right) x^n = 0$$

so that:

$$c_n = -\frac{c_{n-1}}{n(2n-1)}$$

Again by taking $c_0 = 1$, we get:

$$c_1 = -\frac{1}{1} = -1, \quad c_2 = -\frac{c_1}{2 \cdot 3} = \frac{1}{6}, \quad c_3 = -\frac{c_2}{3 \cdot 5} = -\frac{1}{90}, \ldots$$

We now have $y_1$ and $y_2$. Putting it all together, the general solution is:

$$y(x) = C_1 y_1 + C_2 y_2$$

or

$$y(x) = C_1 x\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + h.o.t\right) + C_2\sqrt{x}\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + h.o.t\right)$$

In Maple, verify this:

```
Order:=4;
de:=2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+(1+x)*y(x)=0;
dsolve(de,y(x),type=series);
```

4. (Problem 16, Sect 6.2) Solve: $2xy'' + 5y' + xy = 0$ about the point $x = 0$.

   We multiply through by $x$, although that is not necessary, to get:

$$2x^2y'' + 5xy' + x^2y = 0$$

Now substitute the power series for $y, y', y''$ to get:

$$2x^2\sum_{n=0}^{\infty} c_n(r+n)(r+n-1)x^{r+n-2} + 5x\sum_{n=0}^{\infty} c_n(r+n)x^{r+n-1} + x^2\sum_{n=0}^{\infty} c_n x^{r+n} = 0$$

Simplifying,

$$\sum_{n=0}^{\infty} 2c_n(r+n)(r+n-1)x^{r+n} + \sum_{n=0}^{\infty} 5c_n(r+n)x^{r+n} + \sum_{n=0}^{\infty} c_n x^{r+n+2} = 0$$

The last sum starts with $x^{r+2}$, so pull the first two terms off the first two sums:

$$\left(2c_0 r(r-1) + 5c_0 r\right)x^r + \left(2c_1(r+1)r + 5c_1(r+1)\right)x^{r+1} +$$

$$\sum_{n=2}^{\infty}(2c_n(r+n)(r+n-1) + 5c_n(r+n) + c_{n-2})x^{r+n} = 0$$

The first term gives the indicial equation,

$$r(2r+3) = 0 \Rightarrow r = 0, r = \frac{-3}{2}$$

The second term implies that $c_1 = 0$ for either $r$, and for the rest of the coefficients,

$$c_n = -\frac{c_{n-2}}{(n+r)(2n+2r+3)}, \quad n = 2, 3, 4, \ldots$$

Choosing $r = -3/2$ first, the formula simplifies to:

$$c_n = -\frac{c_{n-2}}{(2n-3)n}$$

Choosing $r = 0$, we get:

$$c_n = -\frac{c_{n-2}}{n(2n+3)}$$

In either case, set $c_0 = 1$, and we get:

$$y = C_1 x^{-3/2}\left(1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 + \ldots\right) + C_2\left(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 + \ldots\right)$$