Summary: Graphical Analysis and Autonomous Differential Equations

1. An *autonomous* D.E. is a D.E. of the form: $y' = f(y)$.

   If we set $y' = 0$, we can determine constants for y such that $f(y) = 0$. These are special solutions, known as *equilibrium solutions*.

   Equilibria can be classified as *stable* (or attracting), *unstable* (or repelling), or *semi-stable*.

2. Because of that form, we are able to construct **two** plots:

   - A Phase Plot, of y' versus y. An analysis of this plot will tell us:
     - What are the equilibrium solutions? [These are where y' touches the $y-$axis].
     - For what values of y is the function $y(t)$ (or $y(x)$) increasing or decreasing? [Look for where y' is positive/negative]
     - For what values of y is $y(t)$ concave up or concave down?
       We can see that:

       $$\frac{dy}{dt} = f(y) \quad \Rightarrow \quad \frac{d^2y}{dt^2} = \frac{d}{dt}(f(y)) = \frac{df}{dy} \cdot \frac{dy}{dt} = \frac{df}{dy} \cdot f(y)$$

       By looking at the graph of $y' = f(y)$, we consider (i) the sign of the local slope, $\frac{df}{dy}$, and (ii) the sign of $f(y)$ (above or below the $y-$axis).
   - A plot in solution space, y versus x (or t).
     - In general, the slope of tangent lines to y must be the same along any horizontal line (that is, y' does not depend on x or t)
     - Place the equilibria as horizontal lines.
     - Use the increasing/decreasing, concave up/down information from the phase plot to draw $y(t)$.

3. An example in detail: Analyze the differential equation

   $$y' = y(a - by), \text{ where } a, b \geq 0$$

   (a) First, the phase plot (of y' versus y). This is an upside down parabola crossing the $y-$axis at $y = 0$ and $y = \frac{a}{b}$.

   (b) Increasing/Decreasing:
   - In Regions A and D, $y' < 0$ so $y(x)$ is decreasing for $y < 0$ and $y > a/b$.
   - In Regions B and C, $y' > 0$ so $y(x)$ is increasing for $0 < y < a/b$.

   (c) Concave Up/Concave Down:
   - In Region A in the figure ($y < 0$), we see that $\frac{df}{dy} > 0$ and $f(y) < 0$. Therefore, $y(x)$ is concave down.
   - In Region B ($0 < y < \frac{a}{2b}$), $\frac{df}{dy} > 0$, but $f(y) > 0$. In this region, $y(x)$ is concave up.
   - In Region C ($\frac{a}{2b} < y < \frac{a}{b}$), $\frac{df}{dy} < 0$ and $f(y) > 0$, so $y(x)$ is concave down.
   - In Region D, $\frac{df}{dy} < 0$ and $f(y) < 0$, so $y(x)$ is concave up.

   (d) Now we transfer this information to the plot of y versus x. We see that the equilibrium $y = 0$ is unstable, and the equilibrium $y = a/b$ is stable. We also get fairly accurate plots by hand. Shown is a graph from Maple, where $a = 3$ and $b = 2$.

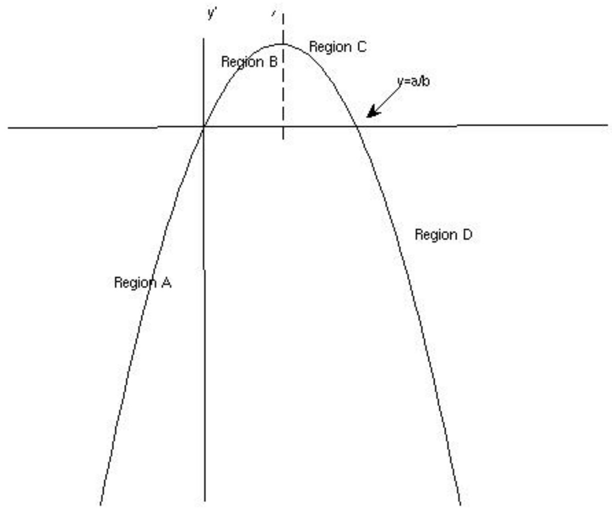

Figure 1: The Phase Plot of $y' = y(a - by)$

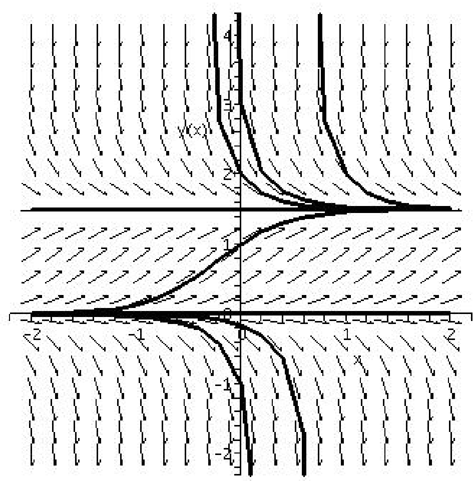

Figure 2: A Sample Direction Field with some sample solutions