Summary: 4.1

Here is what we need to know: Given a linear operator L for a differential equation (assume n^{th} order):

- If $y_1, \ldots, y_k$ are solutions to $L(y) = 0$, so is any linear combination, $c_1y_1 + \ldots + c_ky_k$. (These functions are in the kernel of L).
- There are exactly n linearly independent solutions to $L(y) = 0$. They form the "fundamental set" in that any solution to the IVP can be written in terms of these.
- The general solution to $L(y) = f_1(x) + \ldots + f_k(x)$ is: $y_h(x) + y_{p_1} + \ldots + y_{p_k}$, where y_h is the general solution to $L(y) = 0$, and $y_{p_i}(x)$ solves $L(y) = f_i(x)$.

In particular, here is what happens when $n = 2$:

1. The Existence and Uniqueness Theorem.

   Let $y'' + p(t)y' + q(t)y = f(t)$, $y(t_0) = y_0$. Then if p, q, f are all continuous for $t \in (a, b)$, (and of course, $t_0 \in (a, b)$), then there exists a unique solution to the differential equation, and this solution persists for all $t \in (a, b)$.

2. Definitions:

   (a) Linear Independence: A set of functions, $\{y_1(t), \ldots, y_k(t)\}$ is said to be linearly independent on the interval $[a, b]$ iff the only solution to:

   $$c_1y_1(t) + \ldots + c_ky_k(t) = 0$$

   (for all $t \in [a, b]$) is: $c_1 = c_2 = \ldots = c_k = 0$

   (b) The Wronskian of y_1 and y_2 is:

   $$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

   Note that this is a function of t.

   (c) The Fundamental Set of Solutions: y_1 and y_2 form a fundamental set of solutions to $y'' + p(t)y' + q(t)y = 0$ if the solution to an aribtrary initial value problem can be written as a linear combination of y_1 and y_2. Thus, ALL solutions are of the form:

   $$c_1y_1 + c_2y_2$$

   if y_1 and y_2 form a fundamental set.

3. Theorem (Linear Independence and the Wronskian, in General)

   If f, g are differentiable on an open interval I and if $W(f, g)(t_0) \neq 0$ for some t_0 in I, then f and g are linearly independent on I.

   However, it is possible in this general case, that the Wronskian is zero for all t in I, and the functions f and g are linearly independent. This is not true in the special case that f, g are two solutions to the second order linear, homogeneous differential equation:

4. Abel's Theorem (Linear Independence and the Wronskian, for Solutions to a Linear Homogeneous Differential Equation).

   Let $y'' + p(t)y' + q(t)y = 0$, and let I be an open interval on which p and q are continuous. Let y_1 and y_2 be solutions. Then the Wronskian of y_1 and y_2 is either identically zero on I, (in which case y_1, y_2 are linearly dependent) or it is never zero on I (in which case y_1, y_2 are linearly independent).