Review Questions: Exam 3

You may assume that $u(t-a)$ is the unit step function (a.k.a. Heaviside function). The modeling questions are at the end.

1. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:
   (a)
   $$f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6-t, & 2 < t \end{cases}$$
   (b)
   $$f(t) = \begin{cases} e^{-t}, & 0 \le t \le 5 \\ -1, & t > 5 \end{cases}$$
2. Check your previous answer by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.
3. Determine the Laplace transform:
   (a) $t^2 e^{-9t}$
   (b) $e^{2t} - t^3 - \sin(5t)$
   (c) $u(t-5)(t-5)^4$
   (d) $e^{3t}\sin(4t)$
   (e) $e^t \delta(t-3)$
   (f) $t^2 u(t-4)$
4. Find the inverse Laplace transform:
   (a) $\dfrac{2s-1}{s^2-4s+6}$
   (b) $\dfrac{s^2+16s+9}{(s+1)(s+3)(s-2)}$
   (c) $\dfrac{7}{(s+3)^3}$
   (d) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$
   (e) $\dfrac{3s-2}{2s^2-16s+10}$
   (f) $\dfrac{e^{-2s}}{s^2+2s-2}$
5. Solve the given initial value problems using Laplace transforms:
   (a) $y'' - 7y' + 10y = 0$, $y(0) = 0$, $y'(0) = -3$
   (b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$
   (c) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$
   (d) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$
   (e) $y'' - 2y' - 3y = u(t-1)$, $y(0) = 0$, $y'(0) = -1$
   (f) $y'' - 4y' + 4y = t^2 e^t$, $y(0) = 0$, $y'(0) = 0$
   (g) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$
   (h) $y'' + y = \displaystyle\sum_{k=1}^{\infty} \delta(t - 2k\pi)$, with initial conditions both 0.

6. Evaluate: $\int_0^\infty \sin(3t)\delta(t - \frac{\pi}{2})\, dt$

7. Evaluate, using Laplace transforms: $\sin(t) * t$

8. Use the table to find an expression for $\mathcal{L}(ty')$. Use this to solve:

$$y'' + 3ty' - 6y = 1, \;\; y(0) = 0, \;\; y'(0) = 0$$

9. Characterize ALL (continuous or not) solutions to $y'' + 4y = u(t-1)$, $y(0) = 1$, $y'(0) = 1$.

10. Define $\delta(t - c)$ as a limit of "regular" functions.

11. If $y'(t) = \delta(t - c)$, what is $y(t)$?

12. **Tank Mixing Problems:** Page 117-118, Exercises 5-8. Page 122, Exercise 16.

13. The following system of D.E.s describes the interaction of a population of predators with a population of prey. (a) Which is the predator, and which is the prey (and why)? (b) Find all the equilibrium solutions.

$$\begin{aligned} \dot{x} &= x(-1 + 3y) \\ \dot{y} &= y(0.4 - 2x) \end{aligned}$$

For the spring/mass equations below, the force of gravity is 32ft/sec^2.

14. A 4-foot spring measures 8 feet long after a mass weighing 8 lbs. is attached to it. The medium through which the mass moves offers a damping force equivalent to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5ft/sec.

15. Let $mx'' + \gamma x' + kx = F_0 \cos(\omega t)$ be our mass/spring model. (i) What are the conditions on m, γ, k and ω so that the system is *resonant*? (ii) True or False, and give explicit (mathematical) reasons why: If there is resistance, then the homogeneous part of the solution will tend to zero. (This was also called the transient part of the solution)

16. Assume no damping, and that a mass weighing two pounds stretches a spring 6 inches. The mass is released from equilibrium with an upward velocity of $1ft/sec$. (a) Write the equation of motion, and solve. (b) Suppose we hit the mass with a hammer at time $a > 0$ (use $\delta(t - a)$). Model this and re-solve using Laplace transforms. (Optional: Find a time a so that when we hit the mass with the hammer, it stops all motion. Note that this would have to be at a time when the mass crosses equilibrium).

17. Let x, y, z be three populations of animals with the following properties:

   (a) In the absence of y, z, the population of x grows exponentially.

   (b) In the absence of x, z, the population of y follows logistic growth.

   (c) In the absence of x, y, the population z declines exponentially.

   (d) Populations x, y compete for the same resources, so each of their populations will decrease in proportion to the number of interactions between them.

   (e) Populations x, y are food for the predator z, so each of their populations will decrease in proportion to the number of interactions between them (assume only $x - z$ and $x - y$ interactions). Also, the population of z will increase in proportion to the number of the interactions (again, consider only $x - z$ and $x - y$).

   Build a system of differential equations that will model how the populations x, y, z will change over time. (Do not solve the system)

18. Epidemic Models of the Onset of Social Activities (EMOSA). Epidemic models have been used to model different sorts of social activity. In this question, we develop a model used for the transition from "virgin" to "nonvirgin"[1]. Let P_m, P_f be the proportion of "nonvirgins" males, females (respectively) in a given heterosexual population (so that $1 - P_m, 1 - P_f$ are the proportions of "virgins"). The model assumes the following:

    (a) The proportion of nonvirgin males changes at a rate proportional to the number of virgin-virgin interactions, and proportional to the number of virgin male- nonvirgin female interactions.

    (b) Similarly, the proportion of nonvirgin females changes at a rate proportional to the number of virgin-virgin interactions and proportional to the number of virgin female - nonvirgin male interactions.

    (i) Build the model that follows from these assumptions. (ii) Determine the equilibrium solutions.

    (Side remark: Its interesting to consider how models of disease might be modified to explain social behaviors such as the ones considered here. These questions are very current in sociology and psychology.)

[1]Modified slightly from: "Social contagion, adolescent sexual behavior, and pregnancy: A nonlinear dynamic EMOSA model, Rodgers, J., Rowe, D., Buster, M. *Developmental Psychology*, 1998 34(5), 1096-1113