Laplace Transforms Review Solutions

1. Compute transforms from the definition:
   (a) \[
   \int_0^2 3e^{-st} \, dt + \int_2^\infty (6-t)e^{-st} \, dt = \frac{3}{s} - \frac{3}{s}e^{-2s} + e^{-2s}\left(\frac{4}{s^2} - \frac{1}{s^2}\right)
   \]
   (b) \[
   \int_0^5 e^{-t}e^{-st} \, dt - \int_5^\infty e^{-st} \, dt = \frac{1-e^{-5(s+1)}}{s+1} - \frac{e^{-5s}}{s}
   \]

2. (a) \[
   f(t) = 3(1 - u(t-2)) + (6-t)u(t-2) \quad \text{or} \quad 3(u(t) - u(t-2)) + (6-t)u(t-2)
   \]
   For the Laplace Transform, use \(f(t-a)u(t-a)\). If \(f(t-2) = 3\), then \(f(t) = 3\). If \(f(t) = 6-t\), then \(f(t) = 6 - (t+2) = -t + 4\).
   (b) \[
   f(t) = e^{-t}(1 - u(t-5)) - u(t-5)
   \]
   For the Laplace transform, if \(f(t-5) = e^{-t}\), then \(f(t) = e^{-(t+5)} = e^{-5}e^{-t}\)

3. Compute transforms (using the table)
   (a) \[
   \frac{2}{(s+9)^2}
   \]
   (b) \[
   \frac{1}{s^2} - \frac{6}{s^4} - \frac{5}{s^8+25}
   \]
   (c) \[
   e^{-5s} \frac{4}{s^5}
   \]
   (d) \[
   \frac{4}{(s-3)^2 + 16}
   \]
   (e) Use \(e^{at}f(t) \rightarrow F(s-a)\), with \(f(t) = \delta(t-3)\), so \(e^{-3(s-1)}\).
   (f) Let \(f(t-4) = t^2\), so \(f(t) = (t+4)^2\): \[
   e^{-4s}\left(\frac{4}{s^2} + \frac{8}{s^2} + \frac{16}{s}\right)
   \]

4. Invert the transforms:
   (a) First rewrite as \[
   \frac{2s-1}{(s-2)^2+2}, \quad \text{so} \quad 2e^{2t}\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}e^{2t}\sin(\sqrt{2}t)
   \]
   (b) Via partial fractions: \(e^{-t} - 3e^{-3t} + 3e^{2t}\)
   (c) \[
   \frac{7}{2}e^{-3t}t^2
   \]
   (d) \(u(t-2)[2e^{-2(t-2)} + 2e^{t-2}]\)
   (e) Rewrite to get: \[
   \frac{3}{2} \left(\frac{(s-4)}{(s-3)^2-11} + \frac{10}{3\sqrt{11}(s-3)^2-11}\right), \quad \text{so we get:} \quad \frac{3}{2}e^{4t}\cosh(\sqrt{11}t) + \frac{5}{\sqrt{11}}e^{4t}\sinh(\sqrt{11}t)
   \]
   (f) Write as \(e^{-2s}H(s)\).
   \[
   H(s) = \frac{1}{s^2 + 2s - 2} = \frac{1}{(s+1)^2-3} = F(s+1)
   \]
   \[
   \text{where } F(s) = 1/(s^2 - 3). \text{ This gives } f(t) = 1/\sqrt{3}\sinh(\sqrt{3}t). \text{ Final answer:}
   \]
   \[
   u(t-2) \cdot \frac{1}{\sqrt{3}}e^{-(t-2)}\sinh(\sqrt{3}(t-2))
   \]

5. Solve the diff. eqn:
   (a) \(e^{2t} - e^{5t}\)
   (b) \(-3e^{-3t} + te^{-3t}\)
(c) You might first write:

\[ Y = \frac{s^3 + 4s + 2}{s^3(s^2 + 2s + 2)} = \frac{3}{2} \frac{1}{s} + \frac{1}{s^2} + \frac{1}{2} \frac{3}{s^2 + 2s + 2} \]

\[ -\frac{3}{2} + t + \frac{1}{2} t^2 - \frac{1}{2} e^{-t} \sin(t) + \frac{3}{2} e^{-t} \cos(t) \]

(d) \[ \frac{10}{13} e^{2t} - \frac{23}{13} \cos(3t) + \frac{15}{13} \sin(3t) \]

(e) Let \( h(t) = \frac{1}{3} + \frac{1}{12} e^{3t} + \frac{1}{4} e^{-t} \) Then the solution is: \( y(t) = u(t-1)h(t-1) - \frac{1}{4} e^{3t} + \frac{1}{4} e^{-t}. \)

(f) We need to invert \( \frac{2}{(s-1)(s-2)^2} = \frac{6}{s-1} + \frac{4}{(s-1)^2} + \frac{2}{(s-1)^3} - \frac{6}{s-2} + \frac{2}{(s-2)^2}, \) which is \( e^{2t}(2t - 6) + e^t(t^2 + 4t + 6) \)

(g) \( \frac{1}{2} \sin(2t) + \frac{1}{2} u(t - \pi/2) \sin(2(t - \frac{\pi}{2})) \)

(h) \( y(t) = \sum_{k=1}^{\infty} u_{2\pi k}(t) \sin(t) \) Note that this is:

\[
y(t) = \begin{cases} 
\sin(t), & 0 \leq t < 2\pi \\
2\sin(t), & 2\pi \leq t < 4\pi \\
3\sin(t), & 4\pi \leq t < 6\pi \\
\vdots & \vdots 
\end{cases}
\]

6. Evaluate: \( \int_0^\infty \sin(3t) \delta(t - \frac{\pi}{2}) \, dt \)

We know that \( \int_{-\infty}^{\infty} \delta(t - c)f(t) \, dt = f(c). \) Therefore, for \( c > 0, \)

\[
\int_0^\infty \sin(3t) \delta(t - \pi/2) \, dt = \sin \left( \frac{3\pi}{2} \right) = -1
\]

7. \( \mathcal{L}(t \sin(t)) = \frac{1}{s^2} \cdot \frac{1}{s^2 + 1} = \frac{1}{s^2} - \frac{1}{s^2 + 1} \) and the inverse laplace of that is \( t - \sin(t). \)

8. Use the table to find an expression for \( \mathcal{L}(ty'). \) Use this to solve:

\( y'' + 3ty' - 6y = 1, \quad y(0) = 0, \quad y'(0) = 0 \)

From the table, \( \mathcal{L}(ty') = -Y(s) - sY'(s). \) Take the Laplace transform, and:

\[
s^2Y(S) + 3(-Y - sY') - 6Y = \frac{1}{s}
\]

\[
\Rightarrow Y' + \left( \frac{s^2 - 9}{-3s} \right) Y = -\frac{1}{3s}
\]

Use the method of the integrating factor:

\[
\int \frac{s^2 - 9}{-3s} \, ds = \int \frac{-1}{3} s + \frac{3}{s} \, ds = -\frac{1}{6} s^2 + 3 \ln(s)
\]

Now the integrating factor is \( e^{(-1/6)s^2 + \ln(s^3)} = s^3 e^{(-1/6)s^2}. \)

\[
(s^3 e^{(-1/6)s^2} Y)' = \frac{1}{3s} s^3 e^{(-1/6)s^2} = \frac{1}{3} s e^{(-1/6)s^2}
\]

so that

\[
s^3 e^{(-1/6)s^2} Y = e^{(-1/6)s^2} \Rightarrow Y(s) = \frac{1}{s^3}
\]

Therefore, \( y(t) = \frac{1}{2} t^2. \)
9. Characterize all solutions: We solve by our old method of getting the homogeneous and particular solutions.

\[ y(t) = \begin{cases} 
\cos(2t) + \frac{1}{2} \sin(2t), & 0 \leq t < 1 \\
c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{4}, & t \geq 1 
\end{cases} \]

10. Define the delta function:

\[ \delta(t - c) = \lim_{h \to 0} d_h(t - c) \]

where

\[ d_h(t - c) = \begin{cases} 
\frac{1}{2h}, & c - h < t < c + h \\
0, & \text{otherwise} 
\end{cases} \]

11. If \( y'(t) = \delta(t - c) \), what is \( y(t) \)?

\[ sY = e^{-cs} \to Y = \frac{e^{-cs}}{s} \to y(t) = u(t - c) \]


13. The following system of D.E.s describes the interaction of a population of predators with a population of prey. (a) Which is the predator, and which is the prey (and why)? (b) Find all the equilibrium solutions.

\[ \begin{align*}
\dot{x} &= x(-1 + 3y) \\
\dot{y} &= y(0.4 - 2x)
\end{align*} \]

The prey is \( y \), predators are \( x \) (Note that in the absence of the other, \( x \) decreases but \( y \) increases). The equilibria are where the two derivatives are equal to zero (at the same time). This is where \( x = 0, y = 0 \) or \( x = 1/5, y = 1/3 \).

For the spring/mass equations below, the force of gravity is 32 ft/sec^2.

14. A 4-foot spring measures 8 feet long after a mass weighing 8 lbs. is attached to it. The medium through which the mass moves offers a damping force equivalent to \( \sqrt{2} \) times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 ft/sec.

For \( mx'' + \gamma x' + kx = 0 \), we have that \( m = \frac{8}{32}, k = 2 \) (from \( mg = kL \Rightarrow 8 = k \cdot 4 \) and \( \gamma = \sqrt{2} \)). The D.E. is then:

\[ \frac{1}{4} x'' + \sqrt{2} x' + 2x = 0, x(0) = 0, x'(0) = 5 \]

and the solution is \( x(t) = 5te^{-\sqrt{2}t} \).

15. Let \( mx'' + \gamma x' + kx = F_0 \cos(\omega t) \) be our mass/spring model. (i) What are the conditions on \( m, \gamma, k \) and \( \omega \) so that the system is resonant? (ii) True or False, and give explicit (mathematical) reasons why: If there is resistance, then the homogeneous part of the solution will tend to zero. (This was also called the transient part of the solution)
(i) For resonance, $\gamma = 0$ and $k = \sqrt{\omega}$. (ii) If $\gamma > 0$, we considered $x'' + 2\lambda x' + \omega^2 x$ (different $\omega$), and looked at the homogeneous solutions. The solutions to the characteristic equation were $-\lambda \pm \sqrt{\lambda^2 - \omega^2}$. This gives solutions in which $e^{-\lambda t}$ can be factored out front, which will cause the solutions to die off as $t \to \infty$.

16. Assume no damping, and that a mass weighing two pounds stretches a spring 6 inches. The mass is released from equilibrium with an upward velocity of 1 ft/sec. (a) Write the equation of motion, and solve. (b) Suppose we hit the mass with a hammer at time $a > 0$ (use $\delta(t-a)$). Model this and re-solve using Laplace transforms. (Optional: Find a time $a$ so that when we hit the mass with the hammer, it stops all motion).

For part (a) and the spring constant, convert 6 inches to 0.5 feet to get: $x'' + 64x = 0$, $x(0) = 0$, $x'(0) = -1$. From this, $x(t) = -\frac{1}{8} \sin(8t)$.

If we hit the spring with a hammer, $x'' + 64x = \delta(t-a) \Rightarrow x(t) = -\frac{1}{8} \sin(8t) + \frac{1}{8} U(t-a) \sin(8(t-a))$

For what time(s) $a$ does $\sin(8(t-a)) = \sin(8t)$? The period of $\sin(8t)$ is $\pi/4$, so if we shift by any multiple of that, it would work. So, $a = n \cdot \frac{\pi}{4}$.

17. Let $x, y, z$ be three populations of animals with the following properties:

(a) In the absence of $y, z$, the population of $x$ grows exponentially.
(b) In the absence of $x, z$, the population of $y$ follows logistic growth.
(c) In the absence of $x, y$, the population $z$ declines exponentially.
(d) Populations $x, y$ compete for the same resources, so each of their populations will decrease in proportion to the number of interactions between them.
(e) Populations $x, y$ are food for the predator $z$, so each of their populations will decrease in proportion to the number of interactions between them (assume only $x - z$ and $x - y$ interactions). Also, the population of $z$ will increase in proportion to the number of the interactions (again, consider only $x - z$ and $x - y$).

Build a system of differential equations that will model how the populations $x, y, z$ will change over time. (Do not solve the system)

\[
\begin{align*}
\dot{x} &= k_1 x - k_5 xy - k_7 xz \\
\dot{y} &= y(k_2 + k_3 y) - k_6 xy - k_8 yz \\
\dot{z} &= -k_4 z + k_9 xz + k_{10} yz
\end{align*}
\]

18. Epidemic Models of the Onset of Social Activities (EMOSA). Epidemic models have been used to model different sorts of social activity. In this question, we develop a model used for the transition from “virgin” to “nonvirgin”\(^1\). Let $P_m$, $P_f$ be the proportion of “nonvirgins” males, females (respectively) in a given heterosexual population (so that $1 - P_m$, $1 - P_f$ are the proportions of “virgins”). The model assumes the following:

(a) The proportion of nonvirgin males changes at a rate proportional to the number of virgin-virgin interactions, and proportional to the number of virgin male - nonvirgin female interactions.

(b) Similarly, the proportion of nonvirgin females changes at a rate proportional to the number of virgin-virgin interactions and proportional to the number of virgin female - nonvirgin male interactions.

(i) Build the model that follows from these assumptions. (ii) Determine the equilibrium solutions.

(Side remark: Its interesting to consider how models of disease might be modified to explain social behaviors such as the ones considered here. These questions are very current in sociology and psychology.)

\[
\begin{align*}
P'_m &= k_1(1 - P_m)(1 - P_f) + k_2(1 - P_m)P_f = (1 - P_m)[k_1(1 - P_f) + k_2P_f] \\
P'_f &= k_3(1 - P_m)(1 - P_f) + k_4(1 - P_f)P_m = (1 - P_f)[k_3(1 - P_m) + k_4P_m]
\end{align*}
\]

The equilibria are at \((1, 1)\) and

\[
\left( \frac{k_1}{k_1 - k_2}, \frac{k_3}{k_3 - k_4} \right)
\]