

Local Linear Analysis of Nonlinear Autonomous DEs

Local linear analysis is the process by which we analyze a nonlinear system of differential equations about its equilibrium solutions (also known as critical points or fixed points). Given a system of nonlinear autonomous DEs:

$$\begin{aligned}x' &= f(x, y) \\ y' &= g(x, y)\end{aligned}$$

we first find the equilibrium solutions by setting the derivatives to zero, then solve simultaneously, the system:

$$\begin{aligned}f(x, y) &= 0 \\ g(x, y) &= 0\end{aligned}$$

Given an equilibrium, say $x = a, y = b$, the linearization of the system at the point (a, b) is:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} f_x(a, b) & f_y(a, b) \\ g_x(a, b) & g_y(a, b) \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix}$$

We can then use the Poincaré Diagram to determine the local behavior. We must use some caution in the case of centers and degenerate nodes, however. Because the linearization is an *approximation* of the true solution, the actual solutions are of a slightly perturbed system. This means that while the linearization gives a center, the true solution may be a center or a spiral (we would use a computer simulation to see what we actually get).

Example 1: Competing Species

We use two assumptions in building what is the classic competing species model:

- In the absence of the other, the rate of change of the population uses a logistic model (the model with the environmental threshold).
- The rate of change of each population decreases at a rate proportional to the number of species-species interactions (that is the competition).

Let $x(t), y(t)$ be the two species. Then these assumptions yield the system of differential equations, where we assume all constants are nonnegative:

$$\begin{aligned}x' &= x(\epsilon_1 - \alpha_1 x) - c_1 xy &= x(\epsilon_1 - \alpha_1 x - c_1 y) \\ y' &= y(\epsilon_2 - \alpha_2 y) - c_2 xy &= y(\epsilon_2 - \alpha_2 y - c_2 x)\end{aligned}$$

Right away we see a few of the equilibria. The origin $(0, 0)$ is an equilibrium. If $x = 0$ (or $y = 0$), the other species has the logistic model,

$$x = 0 \Rightarrow y' = y(\epsilon_2 - \alpha_2 y) \quad \text{or} \quad y = 0 \Rightarrow x' = x(\epsilon_1 - \alpha_1 x)$$

From the analysis we did in Chapter 3, we know that the origin will be UNSTABLE, and the other equilibrium, $x = \epsilon_1/\alpha_1$, or $y = \epsilon_2/\alpha_2$ are stable (this is just considering the coordinate axes themselves).

There is one last possible equilibrium- That is, where these lines intersect:

$$\begin{aligned} 0 &= \epsilon_1 - \alpha_1 x - c_1 y & y &= -\frac{\alpha_1}{c_1}x + \frac{\epsilon_1}{c_1} \\ 0 &= \epsilon_2 - \alpha_2 y - c_2 x & y &= -\frac{c_2}{\alpha_2}x + \frac{\epsilon_2}{\alpha_2} \end{aligned}$$

As it will turn out, how these lines intersect will be the key in understanding the behavior of the competing species model. First, some specific examples:

Example 1:

$$\begin{aligned} x' &= x(1 - x - y) = x - x^2 - xy \\ y' &= y(0.75 - y - 0.5x) = 0.75y - y^2 - 0.5xy \end{aligned}$$

Let's perform the usual linearization. The critical points are:

$$(0, 0), (1, 0), (0, 3/4)$$

And the intersection of the lines gives the point $(1/2, 1/2)$. The matrix of partial derivatives is:

$$\begin{bmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{bmatrix}$$

Evaluating this at each of the critical points (in order) gives us:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 \\ 0 & 0.25 \end{bmatrix} \quad \begin{bmatrix} 0.25 & 0 \\ -0.375 & -0.75 \end{bmatrix} \quad \begin{bmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{bmatrix}$$

Using the Poincaré Diagram, we see that the origin is indeed a SOURCE, the equilibria on the x - and y - axes are SADDLES, and the point of intersection of the two lines is a SINK. Putting these together, we can look at the direction field to examine the global behavior. From this we see that if both of the initial populations are not zero, the model predicts that all solutions will tend to the sink at $(1/2, 1/2)$ - We might call this "Peaceful Coexistence".

Example 2:

We now repeat the analysis for a slightly different system:

$$\begin{aligned} x' &= x(1 - x - y) = x - x^2 - xy \\ y' &= y(0.5 - 0.25y - 0.75x) = 0.5y - 0.25y^2 - 0.75xy \end{aligned}$$

The critical points have changed slightly:

$$(0, 0), (1, 0), (0, 2)$$

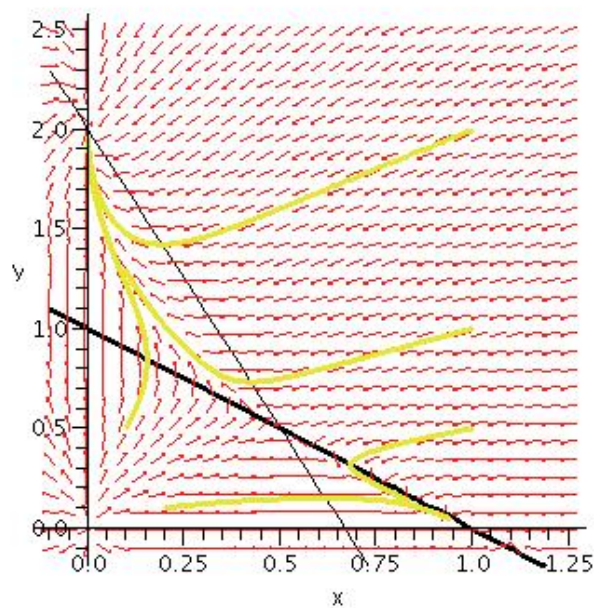
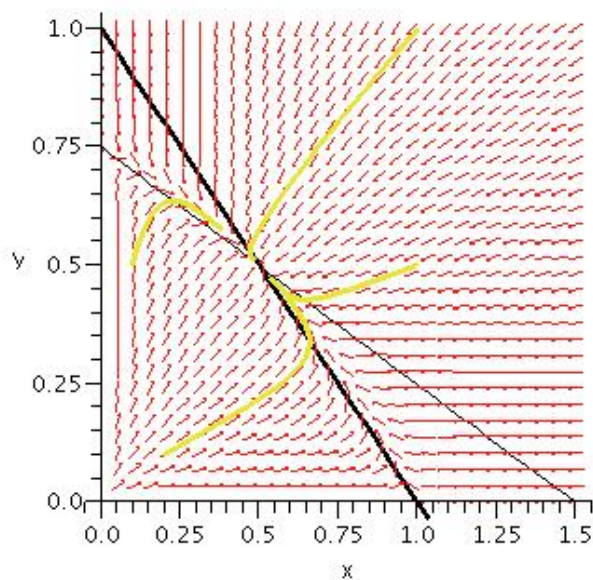


Figure 1: The Competing Species model. The direction field for the first example is the top graph. The lines we see are the nullclines and are NOT solution curves- the thick line is where $x' = 0$, the thin line is where $y' = 0$. In the first example (top graph), we see “Peaceful Coexistence”, where all nonzero initial conditions take us to the equilibrium $(1/2, 1/2)$. In the second model, the two lines have reversed position, and that made the point of intersection a SADDLE instead of a SINK. Now we see that using just about any initial condition, one species will always die off.

And the intersection of the lines still gives the point $(1/2, 1/2)$. The matrix of partial derivatives is:

$$\begin{bmatrix} 1 - 2x - y & -x \\ -.75y & 0.5 - 0.5y - 0.75x \end{bmatrix}$$

Evaluating this at each of the critical points (in order) gives us:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 \\ 0 & -0.25 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ -1.5 & -.5 \end{bmatrix} \quad \begin{bmatrix} -.5 & -.5 \\ -.375 & -.125 \end{bmatrix}$$

Using the Poincaré Diagram, we see that the origin is still a SOURCE, the equilibria on the x - and y - axes are still SADDLES. The big difference is that the point of intersection of the two lines now produces a SADDLE. We will recall that a saddle point is UNSTABLE- this means that for just about any initial condition, one of the two species will die off.

Could we predict which situation we would have had? Yes- Using some algebra, we could show that it depends on the sign of

$$\alpha_1\alpha_2 - c_1c_2$$

We could interpret this as one measure of competition. If that quantity is positive, competition is weak, and you get coexistence. If the quantity is negative, competition is strong, and one species will die off.

Predator Prey

The assumptions used here are much like the ones used in the Competing Species model. Here are the general equations (to the left), and our specific example (to the right):

$$\begin{array}{llll} x' & = & ax - bxy & = x(a - by) & x' & = & x - 0.5xy & = x(1 - 0.5y) \\ y' & = & -cy + kxy & = y(-c + kx) & y' & = & -0.75y + 0.25xy & = y(-0.75 + 0.25x) \end{array}$$

Solving for the equilibria, from the first equation we get

- $x = 0$: From the second, we must have $y = 0$.
- $y = 2$: From the second, $x = 3$.

We have only two equilibria for this system, $(0, 0)$ and $(3, 2)$.

The linearization in general gives:

$$\begin{bmatrix} f_x(a, b) & f_y(a, b) \\ g_x(a, b) & g_y(a, b) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{bmatrix}$$

Linearizing about the two equilibrium gives (in order):

$$\begin{bmatrix} 1 & 0 \\ 0 & -0.75 \end{bmatrix} \quad \begin{bmatrix} 0 & -1.5 \\ 0.5 & 0 \end{bmatrix}$$

In the first case, we have a SADDLE at the origin (which is what we could have expected). At the point $(3, 2)$ we have a CENTER. We check the direction field to verify this, and we have also plotted the individual solution, $x(t)$ and $y(t)$ versus t .

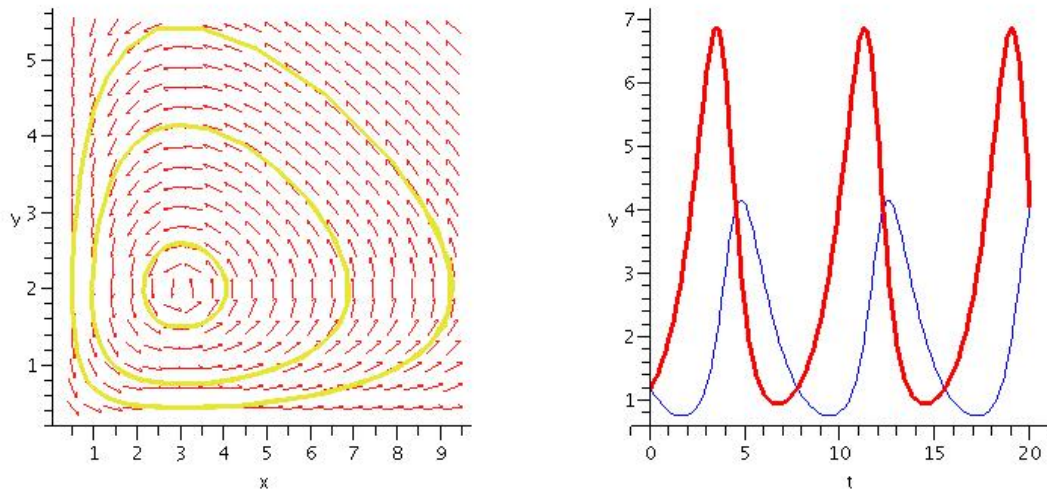


Figure 2: The direction field (top) and a plot of $x(t)$ and $y(t)$ as functions of time (bottom). They both reveal the existence of periodic solutions.

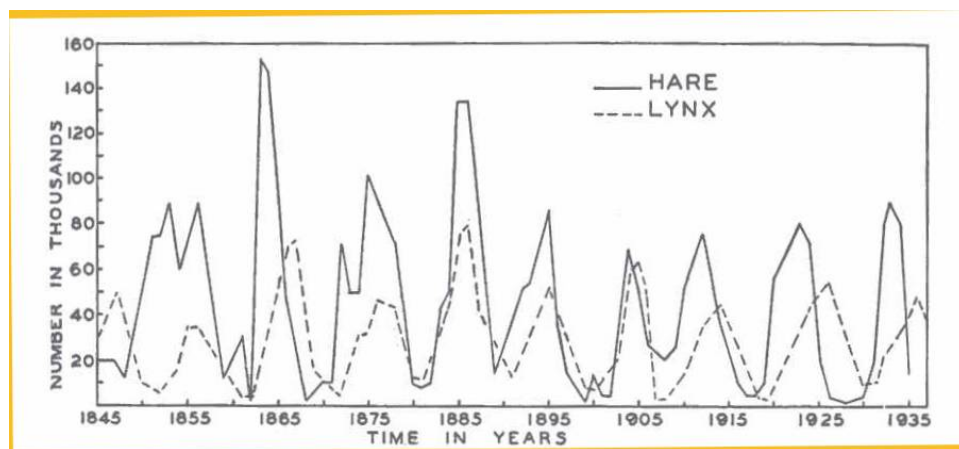


Figure 3: From Odum's "Fundamentals of Ecology". He says that this is taken from an earlier source which is not widely available. Some authors caution that this is actually a composite of several time series, and does not reflect actual populations of rabbits and lynxes, but rather the number of pelts turn in to Hudson's Bay. Is there a predator-prey interaction at work here?

The Lorenz Equations

As a fun example of a three dimensional system, consider the Lorenz equations:

$$\begin{aligned}x' &= -10x + 10y \\y' &= 28x - y - xz \\z' &= -(8/3)z + xy\end{aligned}$$

Following our usual mode of analysis, first we find the equilibrium solutions. From the first equation, $x = y$, which we substitute into the last two, giving:

$$\begin{aligned}0 &= 27x - xz \\0 &= -(8/3)z + x^2\end{aligned}$$

Using the second equation, $x(27 - z) = 0$, then $x = 0$ or $z = 27$. Putting these into the third equation,

- If $x = 0$, then $z = 0$. One equilibrium is $(0, 0, 0)$.
- If $z = 27$, then

$$x^2 = \frac{216}{3} = 72 \quad x = \pm 6\sqrt{2}$$

and we have two equilibrium solutions:

$$(6\sqrt{2}, 6\sqrt{2}, 27) \quad (-6\sqrt{2}, -6\sqrt{2}, 27)$$

We now linearize the system by examining the matrix of partial derivatives:

$$\begin{bmatrix} -10 & 10 & 0 \\ 28 - z & -1 & -x \\ y & x & -8/3 \end{bmatrix}$$

At the origin, we compute the matrix corresponding to the local linear system. To help us analyze the behavior, we will also include the eigenvalues/vectors of the matrix:

$$\begin{aligned}& \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix} \\ \lambda_1 &= 11.8, \quad \mathbf{v}_1 = \begin{bmatrix} .46 \\ 1 \\ 0 \end{bmatrix} \quad \lambda_2 = -22.9, \quad \mathbf{v}_2 = \begin{bmatrix} -.78 \\ 1 \\ 0 \end{bmatrix} \\ \lambda_3 &= -8/3 \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

We interpret this as saying that, in the $x-y$ plane, there is a saddle- One direction is attracting, the other is repelling. There is also a downward attracting force in the z -component. Notice that in a three-dimensional saddle, we can have several different ways of mixing attracting and repelling directions.

At the other two equilibria, we have:

$$\begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -6\sqrt{2} \\ 6\sqrt{2} & 6\sqrt{2} & -8/3 \end{bmatrix} \quad \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & 6\sqrt{2} \\ -6\sqrt{2} & -6\sqrt{2} & -8/3 \end{bmatrix}$$

In both cases, we have the following eigenvalues (I've left off the eigenvectors- We might see them in the direction field):

$$\lambda_1 = -13.85457791, \quad \lambda_{2,3} = .094 \pm 10.19 i$$

We interpret this to mean that there is one attracting direction, and a (weak) spiral source at each of these equilibria.

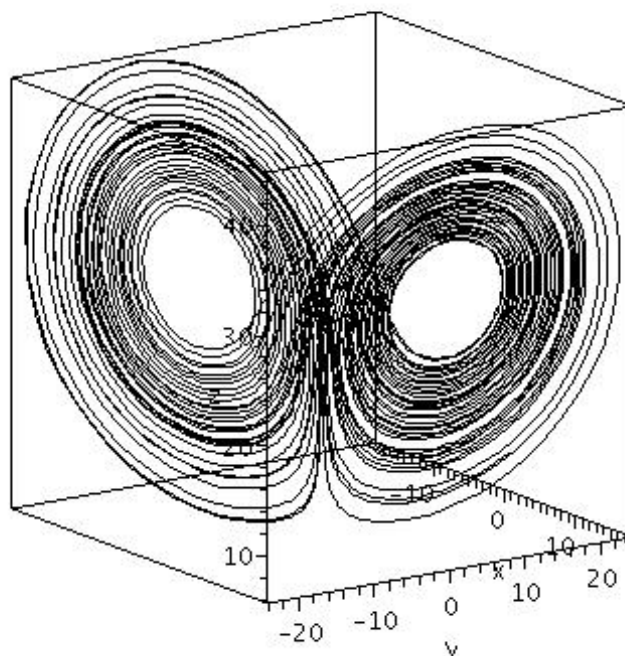


Figure 4: A computed solution to the Lorenz Equations. Notice the appearance of the two weak spiral sources. If you were to run another simulation from an arbitrary starting point, you would get something very similar to this (once the solution got close to this region, and if you don't start at an equilibrium!).

Exercises:

Here is a set of review exercises for all of the things we've covered since looking at Power Series solutions:

1. Convert each differential equation below into an equivalent system of first order differential equations.

(a) $y'' + 3y' + 2y = 0$

(b) $y''' = 3y'' - y + t$

(c) $y'' + yy' = 0$

2. Convert each of the systems $\mathbf{x}' = A\mathbf{x}$ into a single second order differential equation, and solve it using methods from Chapter 3:

(a)

$$A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

3. Suppose we have a system of two differential equations, and the system gave us the eigenvalues and eigenvectors listed. For each, write the general solution to the differential equation:

(a)

$$\lambda_1 = -2 \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \lambda_2 = -1 \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(b)

$$\lambda = -1 + i \quad \mathbf{v} = \begin{bmatrix} 1 + i \\ 2 \end{bmatrix}$$

(c)

$$\lambda = 1, 1 \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

where \mathbf{w} is the generalized eigenvector.

4. For each matrix, find the eigenvalues and eigenvectors. If there is only one eigenvector, find the associated generalized eigenvector.

(a)

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}$$

(d)

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

5. For each system below, find y as a function of x by first writing the differential equation as dy/dx .

(a)

$$\begin{aligned} x' &= -2x \\ y' &= y \end{aligned}$$

(c)

$$\begin{aligned} x' &= -(2x + 3) \\ y' &= 2y - 2 \end{aligned}$$

(b)

$$\begin{aligned} x' &= y + x^3y \\ y' &= x^2 \end{aligned}$$

(d)

$$\begin{aligned} x' &= -2y \\ y' &= 2x \end{aligned}$$

6. Suppose that $\mathbf{x}' = A\mathbf{x}$, where the matrix A is given below. Using the Poincaré Diagram, state how the classification of the equilibrium changes with α .

(a)

$$A = \begin{bmatrix} 2 & \alpha \\ 3 & 2 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} \alpha & 2 \\ \alpha + 2 & 2 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} \alpha & 2 \\ \alpha + 2 & 2 \end{bmatrix}$$

7. For each system below: Find the equilibrium solutions, and linearize about each of them. Finally, use the Poincaré Diagram to classify each of the equilibrium solutions.

Once you're ready to do the predictions, check the Maple Worksheet online.

(a) Coexistence or Extinction?

$$\begin{aligned} x' &= x \left(\frac{3}{2} - x - \frac{1}{2}y \right) \\ y' &= y \left(2 - y - \frac{3}{4}x \right) \end{aligned}$$

(c) Predation or Competition (Explain)?

What happens at $t \rightarrow \infty$?

$$\begin{aligned} x' &= x \left(1 - \frac{1}{2}x - \frac{1}{2}y \right) \\ y' &= y \left(-\frac{1}{4} + \frac{1}{2}x \right) \end{aligned}$$

(b) In this case, try getting the eigenvalues and eigenvectors for the saddles, and try to plot the result!

$$\begin{aligned} x' &= \cos(y) \\ y' &= \sin(x) \end{aligned}$$

(d) Coexistence or Extinction?

$$\begin{aligned} x' &= x(1 - x - y) \\ y' &= y \left(\frac{3}{2} - y - x \right) \end{aligned}$$