Exercises:

Here is a set of review exercises for all of the things we've covered since looking at Power Series solutions:

1. Convert each differential equation below into an equivalent system of first order differential equations.

   (a) $y'' + 3y' + 2y = 0$. Let $x_1 = y, x_2 = y'$. Then:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 3x_2 \end{aligned}$$

   (b) $y''' = 3y'' - y + t$. Let $x_1 = y, x_2 = y', y_3 = y''$. Then:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 3x_3 - x_1 + t \end{aligned}$$

   (c) $y'' + yy' = 0$. Let $x_1 = y, x_2 = y'$. Then:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 x_2 \end{aligned}$$

2. Convert each of the systems $\mathbf{x}' = A\mathbf{x}$ into a single second order differential equation, and solve it using methods from Chapter 3:

   (a)

   $$A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \Rightarrow \quad \begin{aligned} x' &= x + 2y \\ y' &= -5x - y \end{aligned} \qquad \Rightarrow y = \frac{1}{2}(x' - x)$$

   Substituting this into the second DE, we get:

   $$\frac{1}{2}(x'' - x) = -5x - \frac{1}{2}(x' - x) \quad \Rightarrow \quad x'' + 9x = 0$$

   Notice that $\text{Tr}(A) = 0$ and $\det(A) = 9$, so we could have gotten this equation directly as:

   $$x'' - \text{Tr}(A)x' + \det(A)x = 0$$

   as we showed in class.
   Anyway, solving the characteristic equation, we get that $r = \pm 3i$, and

   $$x(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

   Using $y = \frac{1}{2}(x' - x)$, compute $x' = 3c_2 \cos(3t) - 3c_1 \sin(3t)$ to get:

   $$y(t) = \frac{3c_2 - c_1}{2} \cos(3t) - \frac{3c_1 + c_2}{2} \sin(3t)$$

Extra: We can connect this solution to the eigenvalue-eigenvector solution. The eigenvalues are the same as the roots to the characteristic equation,

$$\lambda = \pm 3i$$

Using $\lambda = 3i$ (your choice), the eigenvector is:

$$\begin{bmatrix} 2 \\ -(1-3i) \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 \\ -\frac{1}{2} + \frac{3}{2}i \end{bmatrix}$$

Now we compute $\mathrm{e}^{\lambda t}\mathbf{v}$ and take the real and imaginary parts:

$$\mathrm{e}^{3ti}\mathbf{v} = \cos(3t) + i\sin(3t) \begin{bmatrix} 1 \\ -\frac{1}{2} + \frac{3}{2}i \end{bmatrix} =$$

$$\begin{bmatrix} \cos(3t) + i\sin(3t) \\ -\frac{1}{2}\cos(3t) - \frac{3}{2}\sin(3t) + i(-\frac{1}{2}\sin(3t) + \frac{3}{2}\cos(3t)) \end{bmatrix}$$

so the solution is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \cos(3t) \\ -\frac{1}{2}\cos(3t) - \frac{3}{2}\sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(3t) \\ -\frac{1}{2}\sin(3t) + \frac{3}{2}\cos(3t) \end{bmatrix}$$

You can verify that this is indeed the exact same solution as before.

(b) Taking the shortcut,

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \quad \Rightarrow \quad x'' - 2x' - 3x = 0$$

with $y = x' - x$. The characteristic equation has solutions

$$r = -1, 3 \Rightarrow x(t) = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{3t}$$

and $y(t) = -2c_1\mathrm{e}^{-t} + 2c_2\mathrm{e}^{3t}$. By the way, we can pretty much read off the eigenvalues and eigenvectors to form the vector version of the solution:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1\mathrm{e}^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\mathrm{e}^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(c) Again taking the shortcut (and use the second equation instead of the first):

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \qquad y'' - 2y' + y = 0$$

where $x = y' + y$. Solving the characteristic equation gives $r = 1, 1$, so that $y = \mathrm{e}^t(c_1 + c_2 t)$. Using the equation $x = y' + y$, we get:

$$x = \mathrm{e}^t\left((2c_1 + c_2) + 2c_2 t\right)$$

Notice that in vector form, we have (see the eigenvector $\mathbf{v}$ and the generalized eigenvector $\mathbf{w}$?)

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^t \left(c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \right)$$

3. Suppose we have a system of two differential equations, and the system gave us the eigenvalues and eigenvectors listed. For each, write the general solution to the differential equation:

(a)

$$\lambda_1 = -2 \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \lambda_2 = -1 \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The solution is:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 = c_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(b)

$$\lambda = -1 + i \quad \mathbf{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

The solution is in general: $\mathbf{x}(t) = c_1 \mathrm{Re}(e^{\lambda t}\mathbf{v}) + c_2 \mathrm{Im}(e^{\lambda t}\mathbf{v})$, so first we compute that:

$$e^{\lambda t}\mathbf{v} = e^{-t}(\cos(t) + i\sin(t)) \begin{bmatrix} 1+i \\ 2 \end{bmatrix} = e^{-t} \begin{bmatrix} \cos(t) - \sin(t) + i(\sin(t) + \cos(t)) \\ 2\cos(t) + i2\sin(t) \end{bmatrix}$$

The solution is therefore:

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} \cos(t) - \sin(t) \\ 2\cos(t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin(t) + \cos(t) \\ 2\sin(t) \end{bmatrix}$$

(c)

$$\lambda = 1, 1 \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

where $\mathbf{w}$ is the generalized eigenvector.
The general solution is $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}(t\mathbf{v} + \mathbf{w})$. Making the substitutions,

$$\mathbf{x}(t) = c_1 e^t e^{-t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^t e^{-t} \left(t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

4. For each matrix, find the eigenvalues and eigenvectors. If there is only one eigenvector, find the associated generalized eigenvector.

(a) Remember that eigenvectors are not unique. You might choose a vector that is a constant multiple of the ones given.

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} \mathrm{Tr}(A) & = 6 \\ \det(A) & = 8 \end{matrix} \quad \Rightarrow \lambda^2 - 6\lambda + 8 = 0 \quad \Rightarrow \lambda = 2, 4$$

- $\lambda = 2$:

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad 3v_1 = v_2 \quad \Rightarrow \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

- $\lambda = 4$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow v_1 = v_2 \quad \Rightarrow \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} \mathrm{Tr}(A) & = -2 \\ \det(A) & = 2 \end{matrix} \quad \Rightarrow \lambda^2 + 2\lambda + 2 = 0 \quad \lambda = -1 \pm i$$

We show the vector for $\lambda = -1+i$. The other eigenvector is the complex conjugate of the one shown.

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} (2-i)v_1 & = & 5v_2 \\ & \text{or} & \\ v_1 & = & (2+i)v_2 \end{matrix} \quad \Rightarrow$$

You have a choice, we choose the bottom one since it looks a little easier to write. We also include the second eigenvector below:

$$\mathbf{v}_1 = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} \mathrm{Tr}(A) & = 0 \\ \det(A) & = 0 \end{matrix} \quad \Rightarrow \lambda^2 = 0 \quad \Rightarrow \lambda = 0, 0$$

$$\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = -3v_2 \quad \Rightarrow \quad \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

For the generalized eigenvector, we solve:

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

In this case, choose any w_1, w_2 that satisfies either equation. For example,

$$-w_1 - 3w_2 = 1 \quad \Rightarrow \quad \mathbf{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(d)

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rl} \mathrm{Tr}(A) & = -4 \\ \det(A) & = 5 \end{array} \quad \Rightarrow \lambda^2 - 4\lambda + 5 = 0 \quad \Rightarrow \lambda = -2 \pm i$$

We show the vector for $\lambda = -2+i$. The other eigenvector is the complex conjugate of the one shown.

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcl} (1+i)v_1 & = & 2v_2 \\ & \text{or} & \\ v_1 & = & -(1-i)v_2 \end{array} \quad \Rightarrow$$

You have a choice, we choose the bottom one since it looks a little easier to write. We also include the second eigenvector below:

$$\mathbf{v}_1 = \begin{bmatrix} 1-i \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$$

5. For each system below, find y as a function of x by first writing the differential equation as dy/dx.

Note: These problems give some practice for the following technique:

$$\begin{array}{rl} x' & = f(x,y) \\ y' & = g(x,y) \end{array} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

(a)

$$\begin{array}{rl} x' & = -2x \\ y' & = y \end{array} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{2x}$$

This equation is both linear and separable. For fun, let's try solving it as a linear differential equation:

$$y' + \frac{1}{2x}y = 0 \quad \Rightarrow \quad \text{Int. Factor: } e^{\frac{1}{2}\int \frac{1}{x}\,dx} = e^{\frac{1}{2}\ln(x)} = \sqrt{x}$$

Multiply by the integrating factor:

$$\sqrt{x}\left(y' + \frac{1}{2x}y\right) = 0 \quad \Rightarrow \quad (y\sqrt{x})' = 0 \quad \Rightarrow \quad y\sqrt{x} = C \quad \Rightarrow \quad y = \frac{C}{\sqrt{x}}$$

(b)

$$\begin{array}{rl} x' & = y + x^3 y \\ y' & = x^2 \end{array} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{y(1+x^3)} \qquad \text{Separable}$$

In this case, it will be fine to leave your answer in implicit form:

$$\int y\,dy = \int \frac{x^2}{1+x^3}\,dx \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{1}{3}\ln|1+x^3| + C$$

(c)

$$\begin{aligned} x' &= -(2x+3) \\ y' &= 2y-2 \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2y-2}{2x+3} \quad \text{Separable and Exact}$$

For fun, let's solve it as an exact equation: $(2y-2)\,dx + (2x+3)\,dy = 0$
Using the first term, we antidifferentiate with respect to x:

$$f(x,y) = 2xy - 2x + g(y) \quad \Rightarrow \quad f_y = 2x + g'(y) \Rightarrow g'(y) = 3$$

so that altogether, the general solution is:

$$2xy - 2x + 3y = C$$

(d)

$$\begin{aligned} x' &= -2y \\ y' &= 2x \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C \quad \Rightarrow x^2 + y^2 = c_2$$

6. Suppose that $\mathbf{x}' = A\mathbf{x}$, where the matrix A is given below. Using the Poincaré Diagram, state how the classification of the equilibrium changes with α.

   The point here is that the classification depends on the three values: Trace, Determinant and Discriminant. Here's how it works:

   (a)

$$A = \begin{bmatrix} 2 & \alpha \\ 3 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \text{Tr}(A) &= 4 \\ \det(A) &= 4 - 3\alpha \\ \Delta &= 12\alpha \end{aligned}$$

   Since the trace is fixed at 4, but the determinant varies through all real numbers, we have a vertical line through the Poincaré diagram at $\text{Tr}(A) = 4$.
   We can see the different types of behavior. For example, to be in Quadrant 4, the determinant is negative, so $\alpha > 4/3$ (in that case, the origin is a SADDLE).
   We get a line of unstable fixed points if $\alpha = 4/3$.
   If $\alpha < 4/3$, we are in Quadrant I. We will have a degenerate source if $\alpha = 0$. On the other two sides of the parabola,

$$0 < \alpha < 4/3 \quad \Rightarrow \quad \text{Origin is Source}$$

   and finally, if $\alpha < 0$, the origin is a SPIRAL SOURCE.

   (b)

$$A = \begin{bmatrix} \alpha & 2 \\ \alpha+2 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \text{Tr}(A) &= 2 + \alpha \\ \det(A) &= 2\alpha - 2(2+\alpha) = -4 \\ \Delta &= (2+\alpha)^2 + 16 \end{aligned}$$

   In this case, since the determinant is fixed at -4, we have a horizontal line through Quadrants III and IV in the Poincaré Diagram. We will always have a SADDLE at the origin.

7. For each system below: Find the equilibrium solutions, and linearize about each of them. Finally, use the Poincaré Diagram to classify each of the equilibrium solutions.

   Once you're ready to do the predictions, check the Maple Worksheet online.

   (a) Coexistence or Extinction?

$$\begin{aligned} x' &= x\left(\tfrac{3}{2} - x - \tfrac{1}{2}y\right) \\ y' &= y\left(2 - y - \tfrac{3}{4}x\right) \end{aligned}$$

   We have 4 equilibrium solutions:

$$(0,0) \quad \left(\frac{3}{2},0\right) \quad (0,2) \quad \left(\frac{4}{5},\frac{7}{5}\right)$$

   The linearization will come from the partial derivatives:

$$\begin{bmatrix} \frac{3}{2} - 2x - \frac{1}{2}y & -\frac{1}{2}x \\ -\frac{3}{4}y & 2 - 2y - \frac{3}{4}x \end{bmatrix}$$

   The four matrices we get are (in the previous order):

$$\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} -\frac{3}{2} & -\frac{3}{4} \\ 0 & \frac{7}{8} \end{bmatrix} \quad \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & -2 \end{bmatrix} \quad \begin{bmatrix} -\frac{4}{5} & -\frac{2}{5} \\ -\frac{21}{20} & -\frac{7}{5} \end{bmatrix}$$

   (SORRY about the fractions!) We get (in the previous order):
   SOURCE, SADDLE, SADDLE, SINK
   Note that to classify, we do not necessarily need to compute the trace, determinant and discriminant exactly, we just need to know if they are positive or negative.
   If the solutions begin somewhere in the region $x > 0, y > 0$, then the population will head towards the sink (Peaceful Coexistence).

   (b) In this case, try getting the eigenvalues and eigenvectors for the saddles, and try to plot the result!

$$\begin{aligned} x' &= \cos(y) \\ y' &= \sin(x) \end{aligned}$$

   See the handwritten solution- Linked right under this document on our class website.

   (c) Predation or Competition (Explain)? What happens at $t \to \infty$?

$$\begin{aligned} x' &= x\left(1 - \tfrac{1}{2}x - \tfrac{1}{2}y\right) \\ y' &= y\left(-\tfrac{1}{4} + \tfrac{1}{2}x\right) \end{aligned}$$

   This is Predation. Notice that in the absence of y, the rate of change of x is logistic growth. Similarly, in the absence of x, the rate of change of y is negative-

Therefore, $x(t)$ is the population of PREY, and $y(t)$ is the population of PREDATOR.
There are three equilibrium solutions:

$$(0,0) \qquad (2,0) \qquad \left(\frac{1}{2},\frac{3}{2}\right)$$

Linearizing, the general matrix of partials is:

$$\begin{bmatrix} 1-x-\frac{1}{2}y & -\frac{1}{2}x \\ \frac{1}{2}y & -\frac{1}{4}+\frac{1}{2}x \end{bmatrix}$$

The three matrices are (in the previous order):

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 \\ 0 & \frac{3}{4} \end{bmatrix} \qquad \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & 0 \end{bmatrix}$$

In order, the equilibria are: SADDLE, SADDLE, SPIRAL SINK.
In this case, for all solutions starting in the region $x > 0$, $y > 0$, they all spiral towards the sink- This represents a balance between predators and prey.

(d) Coexistence or Extinction?

$$\begin{aligned} x' &= x(1-x-y) \\ y' &= y\left(\tfrac{3}{2}-y-x\right) \end{aligned}$$

The equilibrium solutions are:

$$(0,0) \qquad \left(0,\frac{3}{2}\right), \qquad (1,0)$$

Interesting to note that our two nullclines (those were the lines $y = -x+1$ and $y = -x+\frac{3}{2}$) do not intersect. Therefore, it seems reasonable that this will force one of the species to die off.
The matrix of partial derivatives is:

$$\begin{bmatrix} 1-2x-y & -x \\ -y & \frac{3}{2}-2y-x \end{bmatrix}$$

And linearizing at each equilibrium gives the following (in order):

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \qquad \begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

The equilibria are (in order): SOURCE, SINK, SADDLE (One of the species will become extinct).