Practice Sheet:
Solving a system of two equations

Occasionally, it will be very convenient to be able to quickly solve two linear equations in two unknowns. For example, solve for C_1, C_2:

$$\begin{aligned} aC_1 + bC_2 &= e \\ cC_1 + dC_2 &= f \end{aligned}$$

A very fast method is known as Cramer's Rule. To use Cramer's rule, we first recall the definition of the determinant (as seen in both Calc III and Linear Algebra):

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: Compute the determinant: $\begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix}$. Solution: $(3)(2) - (-2)(1) = 6 + 2 = 8$.

Cramer's Rule:

The solution to the system of equations:

$$\begin{aligned} aC_1 + bC_2 &= e \\ cC_1 + dC_2 &= f \end{aligned}$$

is $C_1 = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \dfrac{ed - bf}{ad - bc} \qquad C_2 = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \dfrac{af - ce}{ad - bc}$, provided $ad - bc \neq 0$.

Example: Solve the system:

$$\begin{aligned} 3C_1 + 2C_2 &= 4 \\ C_1 - 3C_2 &= 2 \end{aligned}$$

The solution is:

$$C_1 = \frac{\begin{vmatrix} 4 & 2 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix}} = \frac{-12 - 4}{-9 - 2} = \frac{16}{11} \qquad C_2 = \frac{\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix}} = \frac{6 - 4}{-9 - 2} = -\frac{2}{11}$$

And we can check the computation:

$$3 \cdot \frac{16}{11} + 2 \cdot \frac{-2}{11} = \frac{44}{11} = 4 \quad \text{and} \quad \frac{16}{11} - 3 \cdot \frac{-2}{11} = \frac{22}{11} = 2$$

Nice! We can immediately compute the solution, no matter how messy the fractions are!

Some special cases: If the denominator is zero, we might have no solution, or an infinite number of solutions:

- Consider the system:

$$\begin{aligned} C_1 + C_2 &= 1 \\ 3C_1 + 3C_2 &= 3 \end{aligned}$$

  We see that the second equation is just a constant multiple of the first. There are an infinite number of solutions of the form: $C_1 + C_2 = 1$.

- Consider almost the same system:

$$\begin{aligned} C_1 + C_2 &= 1 \\ 3C_1 + 3C_2 &= 1 \end{aligned}$$

  We can think of these as parallel lines in the $C_1 - C_2$ plane- There is no solution to this system.

Practice Problems:

Solve the following systems:

1. $$\begin{aligned} C_1 + C_2 &= 2 \\ -2C_1 - 3C_2 &= 3 \end{aligned}$$

2. Try this one in abstract form: $$\begin{aligned} C_1 + C_2 &= y_0 \\ r_1C_1 + r_2C_2 &= v_0 \end{aligned}$$

3. $$\begin{aligned} C_1 + C_2 &= 2 \\ 3C_1 + C_2 &= 1 \end{aligned}$$

4. $$\begin{aligned} 2C_1 - 5C2 &= 3 \\ 6C_1 - 15C_2 &= 10 \end{aligned}$$

5. $$\begin{aligned} 2x - 3y &= -1 \\ 3x - 2y &= 1 \end{aligned}$$