Hyperbolic Functions- Solutions

Definition: The hyperbolic sine, denoted by $\sinh(x)$, is defined as:

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

Similarly, the hyperbolic cosine is defined:

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

Practice questions with the hyperbolic functions:

1. Plot the hyperbolic sine and cosine. What do they look like? Are they periodic functions?

   From Maple, see Figure 1 (left function is the hyperbolic sine). They are NOT periodic.

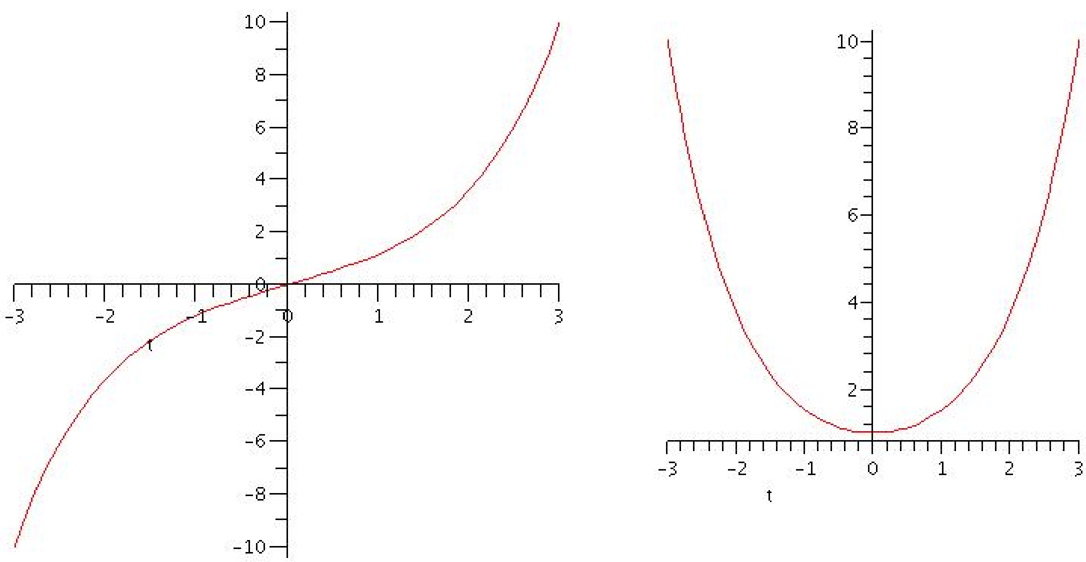

Figure 1: Graphs of the Hyperbolic Sine (left) and Cosine (right)

2. Show, using the definitions, that the hyperbolic sine is an odd function[1] and the hyperbolic cosine is even.

   The hyperbolic sine is odd:

$$\sinh(-x) = \frac{1}{2}\left(e^{-x} - e^{x}\right) = -\frac{1}{2}\left(e^{x} - e^{-x}\right) = -\sinh(x)$$

[1] Recall that f is odd if $f(-x) = -f(x)$, and that f is even if $f(-x) = f(x)$.

The hyperbolic cosine is even:

$$\cosh(-x) = \frac{1}{2}\left(\mathrm{e}^{-x} + \mathrm{e}^{x}\right) = \frac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) = \cosh(x)$$

3. Show, using the definitions, that:

$$\cosh^2(x) - \sinh^2(x) = 1$$

(Don't confuse this with the Pythagorean Identity: $\cos^2(x) + \sin^2(x) = 1$)

Go ahead and just compute:

$$\cosh^2(x) = \frac{1}{4}(\mathrm{e}^{x} + \mathrm{e}^{-x})^2 = \frac{1}{4}\left(\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}\right)$$

Similarly,

$$\sinh^2(x) = \frac{1}{4}(\mathrm{e}^{x} - \mathrm{e}^{-x})^2 = \frac{1}{4}\left(\mathrm{e}^{2x} - 2 + \mathrm{e}^{-2x}\right)$$

From there, just subtract to get 1.

4. Show, using the definitions, that:

$$\frac{d}{dx}(\sinh(x)) = \cosh(x) \qquad \text{and} \qquad \frac{d}{dx}(\cosh(x)) = \sinh(x)$$

(Don't confuse these with the derivatives of sine and cosine!)

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{1}{2}\mathrm{e}^{x} - \frac{1}{2}\mathrm{e}^{-x}\right) = \frac{1}{2}\mathrm{e}^{x} + \frac{1}{2}\mathrm{e}^{-x} = \cosh(x)$$

And,

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\left(\frac{1}{2}\mathrm{e}^{x} + \frac{1}{2}\mathrm{e}^{-x}\right) = \frac{1}{2}\mathrm{e}^{x} - \frac{1}{2}\mathrm{e}^{-x} = \sinh(x)$$

5. Show that any function of the form:

$$y = A\sinh(mt) + B\cosh(mt)$$

satisfies the differential equation: $y'' = m^2 y$.

From what we did in Part 4:

$$y = A\sinh(mt) + B\cosh(mt) \quad \Rightarrow \quad \frac{dy}{dt} = Am\cosh(mt) + Bm\sinh(mt)$$

and

$$y'' = Am^2\sinh(mt) + Bm^2\cosh(mt) = m^2(A\sinh(mt) + B\cosh(mt))$$

6. Show that any function of the form:

$$y = A\sin(\omega t) + B\cos(\omega t)$$

satisfies the differential equation: $y'' = -\omega^2 y$

For this one,

$$y' = A\omega\cos(\omega t) - B\omega\sin(\omega t)$$

and

$$y'' = -A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t)$$

so that

$$y'' = -\omega^2\left(A\sin(\omega t) + B\cos(\omega t)\right) = -\omega^2 y$$