Integration By Parts- Via a Table

Typically, integration by parts is introduced as:

\[ \int u \, dv = uv - \int v \, du \]

We want to be able to compute an integral using this method, but in a more efficient way. Consider the following table:

\[ \int u \, dv \quad \Rightarrow \quad \begin{array}{c|c|c}
+ & u & \frac{dv}{v} \\
- & \frac{du}{u} & v \\
\end{array} \]

The first column switches \( \pm \) signs, the second column differentiates \( u \), and the third column antidifferentiates \( dv \). We can write the result of integration as multiplying the sign, +1 times \( u \) then going down along a diagonal and multiplying by \( v \). We then add the integral of the product going straight across.

Using this table, we can perform multiple integration by parts at one time. Consider this example, with the corresponding table:

\[ \int t^2 e^{-3t} \, dt \quad \Rightarrow \quad \begin{array}{c|c|c|c}
+ & t^2 & e^{-3t} \\
- & 2t & (-1/3)e^{-3t} \\
+ & 2 & (1/9)e^{-3t} \\
- & 0 & (-1/27)e^{-3t} \\
\end{array} \]

Using the same pattern as before, but continuing through, we see that evidently:

\[ \int t^2 e^{-3t} \, dt = t^2(-1/3)e^{-3t} + (-2t)(1/9)e^{-3t} + 2(-1/27)e^{-3t} + \]

\[ + \int (-0 \cdot (-1/27)e^{-3t} \, dt \]

Simplifying:

\[ \int t^2 e^{-3t} \, dt = -e^{-3t} \left( \frac{1}{3}t^2 + \frac{2}{9}t + \frac{2}{27} \right) \]

Here are a couple more examples that usually require integration by parts:

\[ \int \ln(x) \, dx \quad \Rightarrow \quad \begin{array}{c|c|c}
+ & \frac{\ln(x)}{x} & 1 \\
- & \frac{1}{1/x} & x \\
\end{array} \]
so that:
\[ \int \ln(x) \, dx = x \ln(x) - \int 1 \, dx = x \ln(x) - x \]

Another example, where we integrate by parts twice to get a similar integral on both sides of the equation:

\[ \int e^{-2t} \sin(3t) \, dt \quad \Rightarrow \quad \frac{1}{2} \sin(3t) + \frac{3}{4} \cos(3t) \quad \frac{e^{-2t}}{(1/4)e^{-2t}} \]

So:
\[ \int e^{-2t} \sin(3t) \, dt = \sin(3t)(-1/2)e^{-2t} - 3 \cos(3t)(1/4)e^{-2t} + \int -9 \sin(3t)(1/4)e^{-2t} \]

Simplifying:
\[ \int e^{-2t} \sin(3t) \, dt = -e^{-2t} \left( \frac{1}{2} \sin(3t) + \frac{3}{4} \cos(3t) \right) - \frac{9}{4} \int e^{-2t} \sin(3t) \, dt \]

Now solve for the integral:
\[ \frac{13}{4} \int e^{-2t} \sin(3t) \, dt = -e^{-2t} \left( \frac{1}{2} \sin(3t) + \frac{3}{4} \cos(3t) \right) \]

To finish, multiply both sides by 4/13.

**Extra Practice:**

Maple Commands are given so you can check your answer!

1. \[ \int \sqrt{x} \ln(x) \, dx \quad \text{int}(\sqrt{x} \times \ln(x), x); \]
2. \[ \int x^2 \cos(3x) \, dx \quad \text{int}(x^2 \times \cos(3x), x); \]
3. \[ \int t^3 e^{-2t} \, dt \quad \text{int}(t^3 \times \exp(-2t), t); \]
4. \[ \int e^{-2t} \sin(2t) \, dt \quad \text{int}(\exp(-2t) \times \sin(2t), t); \]
5. \[ \int \tan^{-1}(1/t) \, dt \quad \text{int}(\arctan(1/t), t); \]