Chapter 3 Sample Questions

These are not meant to be comprehensive, they are meant to give you questions out of the context of any particular section of the textbook. Be sure you understand the homework problems and quiz questions.
The following trig formula will be given to you:

$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$$

where $R = \sqrt{A^2 + B^2}$, $tan(\delta) = B/A$. I will not ask you to compute using Equation (13), p. 212 (Cosine differences).

1. Finish the definition: Functions f, g are linearly independent if:

2. If the $W(y_1, y_2) = t^2$, can y_1, y_2 be two independent solutions to $y'' + p(t)y' + q(t)y = 0$? Explain.

3. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

4. Find the solution to the initial value problem:

$$u'' + u = \begin{cases} 3t & \text{if } 0 \le t \le \pi \\ 3(2\pi - t) & \text{if } \pi < t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases} \qquad u(0) = 0 \quad u'(0) = 0$$

5. Solve:

$$u'' + \omega_0^2 u = F_0\cos(\omega t), \quad u(0) = 0 \quad u'(0) = 0$$

   if $\omega \neq \omega_0$. (Hint: Probably easiest to use the Method of Undetermined Coefficients)

6. In class, we said that given:

$$u'' + \omega_0^2 u = F_0\cos(\omega t) \qquad u(0) = 0 \quad u'(0) = 0$$

   If $\omega \neq \omega_0$, then

$$u(t) = \frac{F_0}{(\omega_0^2 - \omega^2)}\left(\cos(\omega t) - \cos(\omega_0 t)\right)$$

   Show the solution if $\omega = \omega_0$ two ways:

   - Start over, with Method of Undetermined Coefficients
   - Take the limit of the above expression as $\omega \to \omega_0$.
   - For extra practice with trig integrals, you might also try to find the solution using Variation of Parameters.

7. On Page 208, we see: "The maximum value of R is:

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \approx \frac{F_0}{\gamma\omega_0}\left(1+\frac{\gamma^2}{8mk}\right)$$

where the last expression is an approximation for small γ."

Assuming that they've found the maximum correctly, show that the approximation is valid for small γ (Hint: Think tangent line)

8. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$, where L is the elongation of the spring due to the mass and g is the acceleration due to gravity.

9. Consider $y'' + p(t)y' + q(t)y = 0$. Show that, if $u(t) + iv(t)$ solves the differential equation, then so must $u(t)$ and $v(t)$ as separate functions. (NOTE: If $a + ib = 0$, then $a = 0$ and $b = 0$).

Side remark: This exercise is a confirmation that, if the solution to the homogeneous equation is $e^{\lambda t}\cos(\mu t) + i\, e^{\lambda t}\sin(\mu t)$, then we can use $e^{\lambda t}\cos(\mu t)$ and $e^{\lambda t}\sin(\mu t)$ as the fundamental set.

10. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

$$t^2 y'' - 2y = 0$$

Find a second linearly independent solution, y_2.

11. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

12. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

    (a) $y'' + 4y' + 4y = t^{-2}e^{-2t}$
    (b) $y'' - 2y' + y = te^t + 4$, $y(0) = 1$, $y'(0) = 1$.
    (c) $y'' + 4y = 3\sin(2t)$, $y(0) = 2$, $y'(0) = -1$.
    (d) $y'' + 9y = \sum_{m=1}^{N} b_m \cos(m\pi t)$

13. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $e^{(3-2i)t}$ (iii) $e^{i\pi}$

14. Find a linear second order differential equation with constant coefficients if

$$y_1 = 1 \qquad y_2 = e^{-t}$$

form a fundamental set, and $y_p(t) = \frac{1}{2}t^2 - t$ is the particular solution.

15. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

16. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.

    If $L(3\mathrm{e}^{2t}) = -9\mathrm{e}^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + \mathrm{e}^{2t}$$

17. Show that, using the substitution $x = \ln(t)$, then the differential equation:

$$4t^2y'' + y = 0$$

    becomes a differential equation with constant coefficients.

    Solve it.

18. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

19. Without using the Wronskian, determine whether $f(x) = x\mathrm{e}^{x+1}$ and $g(x) = (4x-5)\mathrm{e}^{x}$ are linearly independent.

20. Given $y'' + p(t)y' + q(t)y = 0$, is it always possible to construct a fundamental set of solutions? (Be specific as to how to do it. You might find the Existence and Uniqueness Theorem useful).