Review Solutions

1. Finish the definition: Functions f, g are linearly independent if:

   the only solution to $k_1 f(t) + k_2 g(t) = 0$ is $k_1 = k_2 = 0$ on an interval I.

2. If the $W(y_1, y_2) = t^2$, can y_1, y_2 be two independent solutions to $y'' + p(t)y' + q(t)y = 0$? Explain.

   They can, if the solution is only valid for $t > 0$ or $t < 0$.

   This is due to Abel's Theorem (actually, a corollary on Pg. 156), which said that, if two functions are solutions to a linear second order equation, then the Wronskian is either always zero or never zero *on the interval for which the solutions are valid.*

3. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

   The operator is found by getting all terms in y to one side of the equation, everything else on the other. In this case, we have:

   $$L(y) = y' - y^2$$

   This is not a linear operator. We can check using the definition:

   $$L(cy) = cy' - c^2y^2 \neq cL(y)$$

   Furthermore,

   $$L(y_1 + y_2) = (y_1' + y_2') - (y_1 + y_2)^2 \neq L(y_1) + L(y_2)$$

4. Find the solution to the initial value problem:

   $$u'' + u = \begin{cases} 3t & \text{if } 0 \le t \le \pi \\ 3(2\pi - t) & \text{if } \pi < t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases} \qquad u(0) = 0 \quad u'(0) = 0$$

   Without regards to the initial conditions, we can solve the three nonhomogeneous equations. In each case, the homogeneous part of the solution is $c_1 \cos(t) + c_2 \sin(t)$.

   - $u'' + u = 3t$. We would start with $y_p = At + B$. Substituting, we get: $At + B = 3$, so $y_p = 3t$. The general solution in this case is:

     $$u(t) = c_1 \cos(t) + c_2 \sin(t) + 3t$$

   - $u'' + u = 6\pi - 3t$. From our previous analysis, the solution is:

     $$u(t) = c_1 \cos(t) + c_2 \sin(t) + 6\pi - 3t$$

- The last part is just the homogeneous equation.

The only thing left is to find c_1, c_2 in each of the three cases so that the overall function u is continuous:

- $u(0) = 0$, $u'(0) = 0 \Rightarrow$

$$u(t) = -3\sin(t) + 3t \qquad 0 \le t \le \pi$$

- $u(\pi) = 3$ and $u'(\pi) = 6$, so:

$$u(t) = 9\sin(t) + (3 - 6\pi)\cos(t) + 6\pi - 3t \qquad \pi < t < 2\pi$$

- $u(2\pi) = 3 - 6\pi$, $u'(2\pi) = 6$:

$$u(t) = 6\sin(t) + (3 - 6\pi)\cos(t) \qquad t \ge 2\pi$$

5. Solve:

$$u'' + \omega_0^2 u = F_0 \cos(\omega t), \quad u(0) = 0 \quad u'(0) = 0$$

if $\omega \neq \omega_0$. (Hint: Probably easiest to use the Method of Undetermined Coefficients)

The homogeneous part of the solution is $c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$. The guess for the particular part is $y_p = A\cos(\omega t) + B\sin(\omega t)$. Substitute y_p into the differential equation and solve:

$$\begin{array}{rl} \omega_0^2(y_p & = A\cos(\omega t) + B\sin(\omega t)) \\ y_p'' & = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) \\ \hline F_0\cos(\omega t) & = A(\omega_0^2 - \omega^2)\cos(\omega t) + B(\omega_0^2 - \omega^2)\sin(\omega t) \end{array}$$

Therefore, $A = \dfrac{F_0}{\omega_0^2 - \omega^2}$, and $B = 0$.

The solution is now:

$$u = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2}\cos(\omega t)$$

Putting in the initial conditions:

$$u(0) = 0 \Rightarrow 0 = c_1 + \frac{F_0}{\omega_0^2 - \omega^2} \Rightarrow c_1 = -\frac{F_0}{\omega_0^2 - \omega^2}$$

And

$$u'(0) = 0 \Rightarrow c_2\omega_0 = 0 \Rightarrow c_2 = 0$$

The solution is:

$$u(t) = \frac{F_0}{\omega_0^2 - \omega^2}\left(\cos(\omega t) - \cos(\omega_0 t)\right)$$

6. In class, we said that given:

$$u'' + \omega_0^2 u = F_0 \cos(\omega t) \qquad u(0) = 0 \quad u'(0) = 0$$

If $\omega \neq \omega_0$, then

$$u(t) = \frac{F_0}{\omega_0^2 - \omega^2} \left(\cos(\omega t) - \cos(\omega_0 t)\right)$$

Show the solution if $\omega = \omega_0$ two ways:

- Start over, with Method of Undetermined Coefficients
  With undetermined coefficients, we take:

  $$y_p = \left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right) t$$

  We multiply by t since the original guess would have been the solution to the homogeneous equation. Take the first and second derivatives (Hint: Keep track of the sine and cosine coefficients):

  $$\begin{aligned} y_p &= At\cos(\omega_0 t) + Bt\sin(\omega_0 t) \\ y_p' &= (A + B\omega_0 t)\cos(\omega_0 t) + (B - A\omega_0 t)\sin(\omega_0 t) \\ y_p'' &= (2B\omega_0 - A\omega_0^2 t)\cos(\omega_0 t) + (-2A\omega_0 - B\omega_0^2)\sin(\omega_0 t) \end{aligned}$$

  Taking $y_p'' + \omega_0^2 y_p$, we get:

  $$F_0\cos(\omega_0 t) = 2B\omega_0\cos(\omega_0 t) - 2A\omega_0\sin(\omega_0 t)$$

  so that $A = 0$, $B = \dfrac{F_0}{2\omega_0}$. Putting the solution together and solving for the coefficients:

  $$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F}{2\omega_0} t\sin(\omega_0 t) \qquad u(0) = 0 \quad u'(0) = 0$$

  we get that $A = 0$ and $B = 0$. Our final answer:

  $$u(t) = \frac{F_0}{2\omega_0} t\sin(\omega_0 t)$$

- Take the limit of the above expression as $\omega \to \omega_0$.
  We can find the function directly by taking the limit (Use L'Hospital's rule, differentiating with respect to ω):

  $$\lim_{\omega\to\omega_0} \frac{F_0(\cos(\omega t) - \cos(\omega_0 t))}{\omega_0^2 - \omega^2} = \lim_{\omega\to\omega_0} \frac{F_0 \cdot t\sin(\omega_0 t)}{2\omega} = \frac{F_0}{2\omega_0} t\sin(\omega_0 t)$$

- For extra practice with trig integrals, you might also try to find the solution using Variation of Parameters.

  With the variation of parameters, $y_1 = \cos(\omega_0 t)$, $y_2 = \sin(\omega_0 t)$, $g(t) = F_0 \cos(\omega_0 t)$, and the Wronskian is ω_0. Using the formulas,

$$u_1' = -\frac{F_0}{\omega_0}\sin(\omega_0 t)\cos(\omega_0 t) \qquad u_2' = \frac{F_0}{\omega_0}\cos^2(\omega_0 t)$$

  For the first integral, use $u = \sin(\omega_0 t)$, $du = \omega_0 \cos(\omega_0 t)\,dt$. For the second integral, use the half angle formula, $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$:

$$u_1 = -\frac{F_0}{2\omega_0^2}\sin^2(\omega_0 t) \qquad u_2 = \frac{F_0}{2\omega_0^2}\sin(\omega_0 t)\cos(\omega_0 t) + \frac{F_0}{2\omega_0}t$$

  so that

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{F_0}{2\omega_0^2}\sin^2(\omega_0 t)\cos(\omega_0 t) + \frac{F_0}{2\omega_0^2}\sin^2(\omega_0 t)\cos(\omega_0 t) + \frac{F_0}{2\omega_0}t\sin(\omega_0 t)$$

  or, as we've gotten earlier, $y_p = \frac{F_0}{2\omega_0}t\sin(\omega_0 t)$

7. On Page 208, we see: "The maximum value of R is:

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \approx \frac{F_0}{\gamma\omega_0}\left(1+\frac{\gamma^2}{8mk}\right)$$

   where the last expression is an approximation for small γ."

   Assuming that they've found the maximum correctly, show that the approximation is valid for small γ (Hint: Think tangent line)

   Notice that the heart of the matter is that we are saying that:

$$(1-x)^{-1/2} \approx 1 + \frac{1}{2}x$$

   when x is small. This is the equation of the tangent line to $f(x) = (1-x)^{-1/2}$ at $x = 0$. The point that the line goes through is $(0, 1)$ and the slope is:

$$f'(x)\big|_{x=0} = \frac{1}{2}(1-x)^{-3/2} = \frac{1}{2}$$

   and the tangent line is: $y - 1 = \frac{1}{2}(x-0)$ or $y = 1 + \frac{1}{2}x$, which is what was claimed.

8. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$, where L is the elongation of the spring due to the mass and g is the acceleration due to gravity.

Undamped motion means that we have:

$$mu'' + ku = 0 \qquad \Rightarrow \qquad r = \pm\sqrt{\frac{k}{m}}i \doteq \pm\mu i$$

so that the homogeneous solution is:

$$u_h(t) = C_1\cos(\mu t) + C_2\sin(\mu t)$$

The period of this function is:

$$\frac{2\pi}{\mu} = 2\pi\sqrt{\frac{m}{k}}$$

From equilibrium, $mg - kL = 0$, we could write $k = mg/L$. Making this substitution,

$$\frac{2\pi}{\mu} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{(mg/L}} = 2\pi\sqrt{\frac{L}{g}}$$

9. Consider $y'' + p(t)y' + q(t)y = 0$. Show that, if $u(t) + iv(t)$ solves the differential equation, then so must $u(t)$ and $v(t)$ as separate functions. (NOTE: If $a + ib = 0$, then $a = 0$ and $b = 0$).

   There are two ways of doing this: Directly by substitution, or by using a linear operator:

   - Operator: $L(y) = y'' + p(t)y' + q(t)y$ is a linear operator. If $u + iv$ solves the differential equation, then: $L(u + iv) = 0$. Since L is linear, $L(u + iv) = L(u) + iL(v)$. Putting these together,

     $$L(u) + iL(v) = 0 \qquad \Rightarrow L(u) = 0 \text{ and } L(v) = 0$$

     so u, v each solve the differential equation separately.
   - By direct substitution:

     $$(u'' + iv'') + p(t)(u' + iv') + q(t)(u + iv) = 0$$

     Rewriting, and grouping terms:

     $$(u'' + q(t)u' + p(t)u) + i(v'' + p(t)v' + q(t)v) = 0$$

     Therefore, $u'' + p(t)u' + q(t)u = 0$ and $v'' + p(t)v' + q(t)v = 0$.

10. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

    $$t^2y'' - 2y = 0$$

    Find a second linearly independent solution, y_2.

First, rewrite the differential equation in standard form:

$$y'' - \frac{2}{t^2}y = 0$$

Then $p(t) = 0$ and $W(y_1, y_2) = Ce^0 = C$. On the other hand, the Wronskian is:

$$W(y_1, y_2) = \frac{1}{t}y_2' + \frac{1}{t^2}y_2$$

Put these together:

$$\frac{1}{t}y_2' + \frac{1}{t^2}y_2 = C \quad y_2' + \frac{1}{t}y_2 = Ct$$

The integrating factor is t,

$$(ty_2)' = Ct^2 \quad \Rightarrow \quad ty_2 = C_1t^3 + C_2 \quad \Rightarrow \quad C_1t^2 + \frac{C_2}{t}$$

Notice that we have *both* parts of the homogeneous solution, $y_1 = \frac{1}{t}$ and $y_2 = t^2$.

11. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

    The model equation can be written as:

    $$0.01u'' + 0.05u' + ku = 0 \quad \Rightarrow \quad u'' + 5u' + \alpha u = 0$$

    where $100k = \alpha$. The solutions depend on the discriminant,

    $$25 - 4\alpha$$

    If this is zero, we have a system that is critically damped. In this case, $k = 4/2500$

    If the discriminant is negative, the system is underdamped. Solving for k, we get that $k > 4/2500$.

12. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

    (a) $y'' + 4y' + 4y = t^{-2}e^{-2t}$

    Using the Variation of Parameters, $y_p = u_1y_1 + u_2y_2$, we have:

    $$y_1 = e^{-2t} \quad y_2 = te^{-2t} \quad g(t) = \frac{e^{-2t}}{t^2}$$

    with a Wronskian of e^{-4t}. You should find that:

    $$u_1' = -\frac{1}{t} \qquad u_2' = \frac{1}{t^2}$$

$$u_1 = -\ln(t) \qquad u_2 = -\frac{1}{t}$$

so $y_p = -\ln(t)\mathrm{e}^{-2t} - \mathrm{e}^{-2t}$. This last term is part of the homogeneous solution, so this simplifies to $-\ln(t)\mathrm{e}^{-2t}$. Now that we have all the parts,

$$y(t) = \mathrm{e}^{-2t}(C_1 + C_2 t) - \ln(t)\mathrm{e}^{-2t}$$

(b) $y'' - 2y' + y = t\mathrm{e}^t + 4$, $y(0) = 1$, $y'(0) = 1$.
With the Method of Undetermined Coefficients, we first get the homogeneous part of the solution,

$$y_h(t) = \mathrm{e}^t(C_1 + C_2 t)$$

Now we construct our ansatz (Multiplied by t after comparing to y_h):

$$g_1 = t\mathrm{e}^t \quad \Rightarrow \quad y_{p_1} = (At + B)\mathrm{e}^t \cdot t^2$$

Substitute this into the differential equation to solve for A, B:

$$y_{p_1} = (At^3 + Bt^2)\mathrm{e}^t \qquad y'_{p_1} = (At^3 + (3A + B)t^2 + 2Bt)\mathrm{e}^t$$

$$y''_{p_1} = (At^3 + (6A + B)t^2 + (6A + 4B)t + 2B)\mathrm{e}^t$$

Forming $y''_{p_1} - 2y'_{p_1} + y_{p_1} = t\mathrm{e}^t$, we should see that $A = \frac{1}{6}$ and $B = 0$, so that $y_{p_1} = \frac{1}{6}t^3\mathrm{e}^t$.
The next one is a lot easier! $y_{p_2} = A$, so $A = 4$, and:

$$y(t) = \mathrm{e}^t(C_1 + C_2 t) + \frac{1}{6}t^3\mathrm{e}^t + 4$$

with $y(0) = 1$, $C_1 = -3$. Solving for C_2 by differentiating should give $C_2 = 4$. The full solution:

$$y(t) = \mathrm{e}^t\left(\frac{1}{6}t^3 + 4t - 3\right) + 4$$

(c) $y'' + 4y = 3\sin(2t)$, $y(0) = 2$, $y'(0) = -1$.
The homogeneous solution is $C_1\cos(2t) + C_2\sin(2t)$. Just for fun, you could try Variation of Parameters. We'll outline the Method of Undetermined Coefficients:

$$y_p = (A\sin(2t) + B\cos(2t))t = At\sin(2t) + Bt\cos(2t)$$

$$y''_p = (-4At - 4B)\sin(2t) + (4A - 4Bt)\cos(2t)$$

taking $y''_p + 4y_p = 3\sin(2t)$, we see that $A = 0$, $B = -\frac{3}{4}$, so the solution is:

$$y = c_1\cos(2t) + c_2\sin(2t) - \frac{3}{4}t\cos(2t)$$

With $y(0) = 2$, $c_1 = 2$. Differentiating to solve for c_2, we find that $c_2 = -1/8$.

(d) $y'' + 9y = \sum_{m=1}^{N} b_m \cos(m\pi t)$

The homogeneous part of the solution is $C_1\cos(3t) + C_2\sin(3t)$. We see that $3 \neq m\pi$ for $m = 1, 2, 3, \ldots$.

The forcing function is a sum of N functions, the m^{th} function is:

$$g_m(t) = b_m \cos(m\pi t) \quad \Rightarrow \quad y_{p_m} = A\cos(m\pi t) + B\sin(m\pi t)$$

Differentiating,

$$y''_{p_m} = -m^2\pi^2 A\cos(m\pi t) - m^2\pi^2 B\sin(m\pi t)$$

so that $y''_{p_m} + 9y_{p_m} = (9 - m^2\pi^2)A\cos(m\pi t) + (9 - m^2\pi^2)B\sin(m\pi t)$.

Solving for the coefficients, we see that $A = b_m/(9 - m^2\pi^2)$ and $B = 0$. Therefore, the full solution is:

$$y(t) = C_1\cos(3t) + C_2\sin(3t) + \sum_{m=1}^{N} \frac{b_m}{9 - m^2\pi^2}\cos(m\pi t)$$

13. Rewrite the expression in the form $a + ib$:

- $2^{i-1} = \mathrm{e}^{ln(2^{i-1})} = \mathrm{e}^{(i-1)\ln(2)} = \mathrm{e}^{-\ln(2)}\mathrm{e}^{i\ln(2)} = \frac{1}{2}\left(\cos(\ln(2)) + i\sin(\ln(2))\right)$
- $\mathrm{e}^{(3-2i)t} = \mathrm{e}^{3t}\mathrm{e}^{-2ti} = \mathrm{e}^{3t}\left(\cos(-2t) + i\sin(-2t)\right) = \mathrm{e}^{3t}\left(\cos(2t) - i\sin(2t)\right)$

  (Recall that cosine is an even function, sine is an odd function).
- $\mathrm{e}^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

14. Find a linear second order differential equation with constant coefficients if

$$y_1 = 1 \qquad y_2 = \mathrm{e}^{-t}$$

form a fundamental set, and $y_p(t) = \frac{1}{2}t^2 - t$ is the particular solution.

The roots to the characteristic equation are $r = 0$ and $r = -1$. The characteristic equation must be $r(r+1) = 0$ (or a constant multiple of that). Therefore, the differential equation is:

$$y'' + y' = 0$$

For $y_p = \frac{1}{2}t^2 - t$ to be the particular solution,

$$y''_p + y'_p = (1) + (t - 1) = t$$

so the full differential equation must be:

$$y'' + y' = t$$

15. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

Write in standard form:

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

The coefficient functions are all continuous on each of three intervals:

$$(-\infty, 0), (0, 4) \text{ and } (4, \infty)$$

Since the initial time is 3, we choose the middle interval, $(0, 4)$.

16. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.

If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

We see that: $L(3e^{2t}) = -9e^{2t}$. By linearity,

$$cL(3e^{2t}) = L(3ce^{2t}) = -9ce^{2t} = e^{2t}$$

so c must be $-1/9$, and

$$L(-\frac{1}{9}3e^{2t}) = L(-\frac{1}{3}e^{2t}) = e^{2t}$$

Similarly,

$$L(t^2 + 3t) = 5t^2 + 3t - 16$$

We need to multiply the right-side of the equation by -2 to get the desired part of our solution, so multiply both sides by -2:

$$-2L(t^2 + 3t) = -10t^2 - 6t + 32$$

By linearity, $-2L(t^2 + 3t) = L(-2t^2 - 6t)$. The particular solution is therefore,

$$y_p(t) = -2t^2 - 6t - \frac{1}{3}e^{2t}$$

17. Show that, using the substitution $x = \ln(t)$, then the differential equation:

$$4t^2y'' + y = 0$$

becomes a differential equation with constant coefficients.

Solve it.

The way the ODE is written now, the derivative is with respect to t. We need to convert it to a derivative in x:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot \frac{1}{t}$$

And the second derivative:

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{1}{t} + \frac{dy}{dx}\left(-\frac{1}{t^2}\right) = \frac{d^2y}{dx^2} \cdot \frac{dx}{dt} \cdot \frac{1}{t} - \frac{dy}{dx} \cdot \frac{1}{t^2}$$

Now defining $y' = \frac{dy}{dx}$, the differential equation becomes:

$$4t^2\left(y'' \cdot \frac{1}{t^2} - y'\frac{1}{t^2}\right) + y = 4y'' - 4y' + y = 0$$

The characteristic equation has solns: $r = \frac{1}{2}, \frac{1}{2}$

$$y(x) = \mathrm{e}^{(1/2)x}\,(C_1 + C_2 x)$$

Back substituting $x = \ln(t)$, we get:

$$y(t) = \sqrt{t}\,(C_1 + C_2 \ln(t))$$

18. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

    The solution is:
    $$y = \left(\frac{2+\alpha}{5}\right)\mathrm{e}^{3t} + \left(\frac{3-\alpha}{5}\right)\mathrm{e}^{-2t}$$
    For the solution to tend to zero, the first constant must be zero, so $\alpha = -2$.

19. Without using the Wronskian, determine whether $f(x) = x\mathrm{e}^{x+1}$ and $g(x) = (4x-5)\mathrm{e}^{x}$ are linearly independent.

    Form the equation that we are solving, and factor/divide out the e^x term:

    $$C_1 x\mathrm{e} + C_2 4x - 5C_2 = 0$$

    This implies that: $-5C_2 = 0$, so C_2 must be zero. The second requirement would be that $\mathrm{e}C_1 + 4C_2 = 0$, but with $C_2 = 0$, then C_1 must be zero.

    The only solution is $C_1 = C_2 = 0$, so the functions are linearly independent.

20. Given $y'' + p(t)y' + q(t)y = 0$, is it always possible to construct a fundamental set of solutions? (Be specific as to how to do it. You might find the Existence and Uniqueness Theorem useful).

If p, q are continuous on an interval I containing t_0, then y_1 is constructed as the (unique) solution to:

$$y'' + p(t)y' + q(t)y = 0 \qquad y(t_0) = 1 \quad y'(t_0) = 0$$

Similarly, y_2 is constructed as the (unique) solution to:

$$y'' + p(t)y' + q(t)y = 0 \qquad y(t_0) = 0 \quad y'(t_0) = 1$$

The initial conditions will force the Wronskian of y_1, y_2 to be nonzero, which gives us a fundamental set of solutions.

(Recall that this is more of a theoretical result rather than a theorem that we actually use to construct the homogeneous solutions).