Chapter 3, Computing Solutions

From the theory, we know that every initial value problem:

$$ay'' + by' + cy = g(t) \qquad y(t_0) = y_0 \quad y'(t_0) = v_0$$

has a solution that can be expressed as:

$$y(t) = c_1 y_1 + c_2 y_2 + y_p$$

where y_1, y_2 form a fundamental set of solutions to the homogeneous equation, and $y_p(t)$ is the (particular) solution to the nonhomogeneous equation.
We first consider the homogeneous ODE:

Solving $ay'' + by' + cy = 0$

Form the associated characteristic equation (built by using $y = \mathrm{e}^{rt}$ as the ansatz):

$$ar^2 + br + c = 0 \qquad \Rightarrow \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that the solutions depend on the discriminant, $b^2 - 4ac$ in the following way (y_h refers to the solution of the homogeneous equation):

- $b^2 - 4ac > 0 \Rightarrow$ two distinct real roots r_1, r_2. The general solution is:

  $$y_h(t) = c_1 \mathrm{e}^{r_1 t} + c_2 \mathrm{e}^{r_2 t}$$

  If $a, b, c > 0$ (as in the Spring-Mass model) we can further say that r_1, r_2 are negative. We would say that this system is OVERDAMPED.

- $b^2 - 4ac = 0 \Rightarrow$ one real root $r = -b/2a$. Then the general solution is:

  $$y_h(t) = \mathrm{e}^{-(b/2a)t}\,(C_1 + C_2 t)$$

  If $a, b, c > 0$ (as in the Spring-Mass model), the exponential term has a negative exponent. In this case (one real root), the system is CRITICALLY DAMPED.

- $b^2 - 4ac < 0 \Rightarrow$ two complex conjugate solutions, $r = \lambda \pm i\mu$. Then the solution is:

  $$y_h(t) = \mathrm{e}^{\lambda t}\,(C_1 \cos(\mu t) + C_2 \sin(\mu t))$$

  If $a, b, c > 0$, then $\lambda < 0$. In the case of complex roots, the system is said to the UNDERDAMPED. If $\lambda = 0$ (this occurs when there is no damping), we get pure periodic motion, with period $2\pi/\mu$.

Solving $y'' + p(t)y' + q(t)y = 0$

Given $y_1(t)$, we can solve for a second linearly independent solution to the homogeneous equation, y_2, by one of two methods:

- By use of the Wronskian: There are two ways to compute this,
  - $W(y_1, y_2) = C\mathrm{e}^{-\int p(t)\,dt}$ (This is from Abel's Theorem)
  - $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

  Therefore, these are equal, and y_2 is the unknown: $y_1 y_2' - y_2 y_1' = C\mathrm{e}^{-\int p(t)\,dt}$

Summarized Example: $y'' + \frac{2}{t}y' - \frac{2}{t^2} = 0$, with $y_1 = t$.

Abel's Theorem: $W(y_1, y_2) = C\mathrm{e}^{-2\ln(t)} = C/t^2$

Wronskian: $ty_2' - y_2$

These should be the same: $ty_2' - y_2 = \frac{C}{t^2}$ is a linear first order equation. Solve it and ignore the constant to get that $y_2 = t^{-2}$.

- By Variation of Parameters (the method that the text uses in Section 3.5), where $y_2 = u_2(t)y_1(t)$. See Example 3, p. 171 for an example.

Solving for the particular solution.

Our two methods were: Method of Undetermined Coefficients and Variation of Parameters.

- Method of Undetermined Coefficients

  This method is motivated by the observation that, a linear operator of the form $L(y) = ay'' + by' + cy$, acting on certain classes of functions, returns the same class. In summary, we have Table 3.6.1, reproduced below:

if $g_i(t)$ is:	The ansatz y_{p_i} is:
$P_n(t)$	$t^s(a_0 + a_1t + \ldots a_nt^n)$
$P_n(t)\mathrm{e}^{\alpha t}$	$t^s\mathrm{e}^{\alpha t}(a_0 + a_1t + \ldots + a_nt^n)$
$P_n(t)\mathrm{e}^{\alpha t}\sin(\mu t)$ or $\cos(\mu t)$	$t^s\mathrm{e}^{\alpha t}\left((a_0 + a_1t + \ldots + a_nt^n)\sin(\mu t)\right.$ $\left. + (b_0 + b_1t + \ldots + b_nt^n)\cos(\mu t)\right)$

  The t^s term comes from an analysis of the homogeneous part of the solution. That is, multiply by t or t^2 so that no term of the ansatz is included as a term of the homogeneous solution.

- Variation of Parameters: Given $y'' + p(t)y' + q(t)y = g(t)$, with y_1, y_2 solutions to the homogeneous equation, we write the ansatz for the particular solution as:

$$y_p = u_1y_1 + u_2y_2$$

  From our analysis, we saw that u_1, u_2 were required to solve:

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= 0 \end{aligned}$$

  From which we get the formulas for u_1' and u_2':

$$u_1' = \frac{-y_2g}{W(y_1, y_2)} \qquad u_2' = \frac{y_1g}{W(y_1, y_2)}$$