Complex Numbers

1 Introduction

1.1 Real or Complex?

Definition: The complex number z is defined as:

$$z = a + bi \tag{1}$$

where a, b are real numbers and $i = \sqrt{-1}$. (Side note: Engineers typically use j instead of i).

Examples:

$$5 + 2i, \quad 3 - \sqrt{2}i, \quad 3, \quad -5i$$

Real numbers are also complex (by taking $b = 0$).

1.2 Visualizing Complex Numbers

A complex number is defined by it's two real numbers. If we have $z = a + bi$, then:
Definition: The *real part* of $a + bi$ is a,

$$\text{Re}(z) = \text{Re}(a + bi) = a$$

The imaginary part of $a + bi$ is b,

$$\text{Im}(z) = \text{Im}(a + bi) = b$$

To visualize a complex number, we can plot it on the plane. The horizontal axis is for the real part, and the vertical axis is for the imaginary part; $a + bi$ is plotted as the point (a, b).

In Figure 1, we can see that it is also possible to represent the point $a + bi$, or (a, b) in polar form, by computing its modulus (or size), and angle (or argument):

$$|z| = \sqrt{a^2 + b^2} \qquad \phi = \arg(z)$$

We have to be a bit careful defining ϕ- Being an angle, it is not uniquely described ($0 = 2\pi = 4\pi$, etc). It is customary to restrict ϕ to be in the interval $(-\pi, \pi]$.

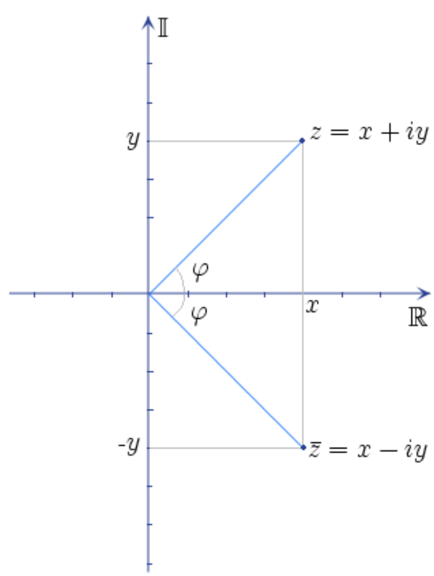

Figure 1: Graphically representing the complex number $z = x + iy$, and visualizing its complex conjugate, $\bar{z}$

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1.3 Operations on Complex Numbers

1.3.1 The Conjugate of a Complex Number

If $z = a + bi$ is a complex number, then its *conjugate*, denoted by $\bar{z}$ is $a - bi$. For example,

$$z = 3 + 5i \Rightarrow \bar{z} = 3 - 5i \qquad z = i \Rightarrow \bar{z} = -i \qquad z = 3 \Rightarrow \bar{z} = 3$$

Graphically, the conjugate of a complex number is it's mirror image across the horizontal axis.

1.3.2 Addition/Subtraction, Multiplication/Division

To add (or subtract) two complex numbers, add (or subtract) the real parts and the imaginary parts separately:

$$(a + bi) \pm (c + di) = (a + c) \pm (b + d)i$$

To multiply, expand it as if you were multiplying polynomials:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

and simplify using $i^2 = -1$. Note what happens when you multiply a number by its conjugate:

$$z\bar{z} = (a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2 = |z|^2$$

Division by complex numbers z, w: $\frac{z}{w}$, is defined by translating it to real number division (rationalize the denominator):

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Example:

$$\frac{1 + 2i}{3 - 5i} = \frac{(1 + 2i)(3 + 5i)}{34} = \frac{-7}{34} + \frac{11}{34}i$$

1.4 The Polar Form of Complex Numbers

1.4.1 Euler's Formula

Any point on the unit circle can be written as $(\cos(\theta), \sin(\theta))$, which corresponds to the complex number $\cos(\theta) + i\sin(\theta)$. It is possible to show the following directly, but we'll use it as a definition:

Definition (Euler's Formula): $\mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta)$.

1.4.2 Polar Form of $a + bi$:

The polar form is defined as:

$$z = r\mathrm{e}^{i\theta} \qquad \text{where} \qquad r = |z| = \sqrt{a^2 + b^2} \qquad \theta = \arg(z)$$

To be sure that the polar form is unique, we restrict θ to be in the interval $(-\pi, \pi]$. You might think of $\arg(z)$ as the four-quadrant inverse tangent- That is:

- If (a, b) is in the first or fourth quadrant, then $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.
- If $a = 0$ and $b \neq 0$, then θ is either $\pi/2$ (for $b > 0$) or $-\pi/2$.
- If (a, b) is in the second quadrant, add π: $\theta = \tan^{-1}\left(\frac{b}{a}\right) + \pi$
- If (a, b) is in the third quadrant, subtract π: $\theta = \tan^{-1}\left(\frac{b}{a}\right) - \pi$
- The argument of zero is not defined.

Best way to remember these: Quickly plot $a + bi$ to see if you need to add or subtract π.

1.5 Exponentials and Logs

The logarithm of a complex number is easy to compute if the number is in polar form:

$$\ln(a+bi) = \ln\left(re^{i\theta}\right) = \ln(r) + \ln\left(e^{i\theta}\right) = \ln(r) + i\theta$$

The logarithm of zero is left undefined (as in the real case). However, we can now compute the log of a negative number:

$$\ln(-1) = \ln\left(1 \cdot e^{i\pi}\right) = i\pi \qquad \text{or the log of } i: \quad \ln(i) = \ln(1) + \frac{\pi}{2}i = \frac{\pi}{2}i$$

Note that the usual rules of exponentiation and logarithms still apply.

To exponentiate a number, we convert it to multiplication (a trick we used in Calculus when dealing with things like x^x):

$$a^b = e^{b\ln(a)}$$

Example, $2^i = e^{i\ln(2)} = \cos(\ln(2)) + i\sin(\ln(2))$

Example: $\sqrt{1+i} = (1+i)^{1/2} = \left(\sqrt{2}e^{i\pi/4}\right)^{1/2} = (2^{1/4})e^{i\pi/8}$

Example: $i^i = e^{i\ln(i)} = e^{i(i\pi/2)} = e^{-\pi/2}$

2 Real Polynomials and Complex Numbers

If $ax^2 + bx + c = 0$, then the solutions come from the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the past, we only took real roots. Now we can use complex roots. For example, the roots of $x^2 + 1 = 0$ are $x = i$ and $x = -i$.

Check:

$$(x-i)(x+i) = x^2 + xi - xi - i^2 = x^2 + 1$$

Some facts about polynomials when we allow complex roots:

1. An n^{th} degree polynomial can always be factored into n roots. (Unlike if we only have real roots!) This is the *Fundamental Theorem of Algebra*.
2. If $a + bi$ is a root to a real polynomial, then $a - bi$ must also be a root. This is sometimes referred to as "roots must come in conjugate pairs".

3 Exercises

1. Suppose the roots to a cubic polynomial are $a = 3$, $b = 1-2i$ and $c = 1+2i$. Compute $(x-a)(x-b)(x-c)$.
2. Find the roots to $x^2 - 2x + 10$. Write them in polar form.
3. Show that:
$$\text{Re}(z) = \frac{z + \bar{z}}{2} \qquad \text{Im}(z) = \frac{z - \bar{z}}{2i}$$
4. For the following, let $z_1 = -3 + 2i$, $z_2 = -4i$
   (a) Compute $z_1\bar{z}_2$, z_2/z_1
   (b) Write z_1 and z_2 in polar form.
5. In each problem, rewrite each of the following in the form $a + bi$:
   (a) e^{1+2i}
   (b) e^{2-3i}
   (c) $e^{i\pi}$
   (d) 2^{1-i}
   (e) $e^{2-\frac{\pi}{2}i}$
   (f) π^i