# Solve $\mathbf{x}' = A\mathbf{x}$

From Section 7.1, we already know how to solve a system of two first order differential equations using the methods of Chapter 3 (after converting to a single second order equation). In these notes, we want to connect the solution to the system with the eigenvalues and eigenvectors of the coefficient matrix $A$.

Given

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{or} \qquad \mathbf{x}' = A\mathbf{x}$$

Inspired by Chapter 3, we use the ansatz:

$$\mathbf{x} = \mathrm{e}^{rt}\mathbf{v} = \mathrm{e}^{rt} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{rt} v_1 \\ \mathrm{e}^{rt} v_2 \end{bmatrix}$$

and substitute it into the differential equation. We will see that $r$ is actually an eigenvalue of $A$, and $\mathbf{v}$ the corresponding eigenvector. First we compute $\mathbf{x}'$, then $A\mathbf{x}$, then we set them equal to each other:

$$\mathbf{x}' = \begin{bmatrix} r\mathrm{e}^{rt} v_1 \\ r\mathrm{e}^{rt} v_2 \end{bmatrix} = r\mathrm{e}^{rt}\mathbf{v}$$

and:

$$A\mathbf{x} = A\mathrm{e}^{rt}\mathbf{v} = \mathrm{e}^{rt}A\mathbf{v}$$

Set these equal to each other:

$$\mathrm{e}^{rt}A\mathbf{v} = r\mathrm{e}^{rt}\mathbf{v}$$

We can divide by $\mathrm{e}^{rt}$ since it is never zero, and we get:

$$A\mathbf{v} = r\mathbf{v}$$

so that $r$ is an eigenvalue of $A$ (we will now stick with the $\lambda$ notation), and $\mathbf{v}$ is the corresponding eigenvector.

This is not the whole story, however. Just as we did in Chapter 3, we will need to find a fundamental set of solutions for our system. And, just as in Chapter 3, we will see that it takes two linearly independent solutions to form that fundamental set.

We will have three cases classified by the eigenvalues (and in Chapter 3 by the roots to the characteristic equation): (i) Two distinct real eigenvalues, (ii) Complex conjugate eigenvalue, and (iii) One eigenvalue, one eigenvector.

## Case 1: Distinct, Real Eigenvalues

Similar to Chapter 3, given two distinct real eigenvalues and their corresponding eigenvectors, the solution to the differential equation is given by:

$$\mathbf{x}(t) = c_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + c_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

**Example:** We solve Problem 10 in Section 7.1:

$$\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \mathbf{x} \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

We first solve for the eigenvalues and eigenvectors. The trace is $1 - 4 = -3$ and the determinant is $-4 + 6 = 2$. The characteristic equation is:

$$\lambda^2 + 3\lambda + 2 = 0$$

which we solve for $\lambda = -2, -1$. The eigenvectors are:

- For $\lambda = -2$, we have the system:

$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{or } 3v_1 - 2v_2 = 0 \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- For $\lambda = -1$, we have:

$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{or } 2v_1 - 2v_2 = 0 \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general solution is then given by:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{-2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

With the initial condition, we have the following system, solved using Cramer's Rule:

$$\begin{aligned} 2c_1 + c_2 &= -1 \\ 3c_1 + c_2 &= 2 \end{aligned} \qquad c_1 = 3 \quad c_2 = -7$$

The solution to the IVP is therefore:

$$\mathbf{x}(t) = 3\mathrm{e}^{-2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 7\mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} x_1(t) &= 6\mathrm{e}^{-2t} - 7\mathrm{e}^{-t} \\ x_2(t) &= 9\mathrm{e}^{-2t} - 7\mathrm{e}^{-t} \end{aligned}$$

## Complex Eigenvalues

We will need to remember Euler's Formula, written with a real part:

$$\mathrm{e}^{(a+bi)t} = \mathrm{e}^{at}\cos(bt) + i \;\; \mathrm{e}^{at}\sin(bt)$$

When we had complex roots to the characteristic equation, we found that we had a linearly independent (real) set of solutions by taking:

$$y(t) = c_1 \mathrm{e}^{at}\cos(bt) + c_2 \mathrm{e}^{at}\sin(bt) = c_1 \mathrm{Re}\left(\mathrm{e}^{(a+bi)t}\right) + c_2 \mathrm{Im}\left(\mathrm{e}^{(a+bi)t}\right)$$

where "Re" and "Im" stand for the real and imaginary parts of the complex number:

$$\mathrm{Re}(a+bi)=a \qquad \mathrm{Im}(a+bi)=b$$

For a system of equations, the notation looks almost identical:

If $\lambda$ is complex with corresponding complex eigenvector $\mathbf{v}$, then the solution to the system is:

$$\mathbf{x}(t)=c_1\mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)+c_2\mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

where $\lambda$ is ONE of the eigenvalues (either one), and $\mathbf{v}$ is its corresponding eigenvector.

**EXAMPLE:**

$$\mathbf{x}'=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\mathbf{x} \qquad \mathbf{x}(0)=\begin{bmatrix}-1\\ 1\end{bmatrix}$$

We first get an eigenvalue and eigenvector:

The trace is 2 and the determinant is 5. The characteristic equation is:

$$\lambda^2-2\lambda+5=0 \qquad (\lambda-1)^2+4=0$$

which we solve for $\lambda$. In this case, completing the square seems faster than the quadratic formula, and: $\lambda=1\pm 2i$. For $\lambda=1+2i$, solve the system on the left to get:

$$\begin{bmatrix}-2i & 2\\ -2 & -2i\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \qquad v_2=iv_1 \qquad \mathbf{v}=\begin{bmatrix}1\\ i\end{bmatrix}$$

To get the fundamental set, we expand the function $\mathrm{e}^{\lambda t}\mathbf{v}$, then we will take the real part and imaginary part of the result:

$$\mathrm{e}^{\lambda t}\mathbf{v}=\mathrm{e}^{(1+2i)t}\begin{bmatrix}1\\ i\end{bmatrix}=\mathrm{e}^t\left(\cos(2t)+i\,\sin(2t)\right)\begin{bmatrix}1\\ i\end{bmatrix}=\mathrm{e}^t\begin{bmatrix}\cos(2t)+i\sin(2t)\\ -\sin(2t)+i\cos(2t)\end{bmatrix}$$

The solution to the differential equation is:

$$\mathbf{x}(t)=c_1\mathrm{e}^t\begin{bmatrix}\cos(2t)\\ -\sin(2t)\end{bmatrix}+c_2\mathrm{e}^t\begin{bmatrix}\sin(2t)\\ \cos(2t)\end{bmatrix}=\begin{bmatrix}\mathrm{e}^t\left(c_1\cos(2t)+c_2\sin(2t)\right)\\ \mathrm{e}^t\left(-c_1\sin(2t)+c_2\cos(2t)\right)\end{bmatrix}$$

After solving for the initial conditions (You should verify this), we get:

$$\mathbf{x}(t)=\begin{bmatrix}\mathrm{e}^t\left(-\cos(2t)+\sin(2t)\right)\\ \mathrm{e}^t\left(\sin(2t)+\cos(2t)\right)\end{bmatrix}$$

## Case 3: One Real Eigenvalue, One Eigenvector

In the rare occurrence that you have one eigenvalue but two eigenvectors (we'll do this in class), go to Case 1. Otherwise, we have the more general case here.

You can read pages 423-424 for more information on this one. This is a special case where we need to find a second eigenvector (called a generalized eigenvector):

- Given an eigenvalue $\lambda$ and eigenvector $\mathbf{v}$, find the "generalized" eigenvector $\mathbf{w}$ by solving the system:

$$\begin{aligned} (a-\lambda)w_1 + bw_2 &= v_1 \\ c\,w_1 + (d-\lambda)w_2 &= v_2 \end{aligned}$$

The solution to the differential equation is then given by:

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}\left(t\mathbf{v} + \mathbf{w}\right)$$

Of course, in this instance we can always use the method of Chapter 3 to solve this, but we want to note the form of the solution before we talk about the geometry in Chapter 9.
**Example:**

$$\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x}$$

The trace is 0 and the determinant is 0. Therefore, $\lambda = 0$ is the only eigenvalue. We now get the eigenvector $\mathbf{v}$:

$$4v_1 - 2v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Now the generalized eigenvector $\mathbf{w}$:

$$\begin{aligned} 4w_1 - 2w_2 &= 2 \\ 8w_1 - 4w_2 &= 4 \end{aligned} \qquad 4w_1 - 2w_2 = 2$$

We take any $w_1, w_2$ that satisfies this relationship- integer solutions are nice (you can change $\mathbf{v}$ if necessary), and in this case we choose $w_1 = 0$ and $w_2 = -1$.

The solution is (in several forms):

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + c_2 \left( t \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 2c_1 + 2c_2 t \\ (4c_1 - c_2) + 4tc_2 \end{bmatrix}$$

We'll check that this is indeed a solution. First, we compute $\mathbf{x}'$ and show that it is equal to $A\mathbf{x}$:

$$\mathbf{x}' = \begin{bmatrix} 2c_2 \\ 4c_2 \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} 2c_1 + 2c_2 t \\ (4c_1 - c_2) + 4tc_2 \end{bmatrix} = \begin{bmatrix} 0 + 2c_2 + 0t \\ 0 + 4c_2 + 0t \end{bmatrix}$$