**Summary: Eigenvalues and Eigenvectors**

Given a system of two expressions in two unknowns, which can be written equivalently as either:

$$\begin{array}{ll} ax & +by \\ cx & +dy \end{array} \qquad \text{or} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{or} \qquad A\mathbf{x}$$

We are interested in finding a scalar $\lambda$ and a non-zero vector $\mathbf{v}$ so that:

$$A\mathbf{v} = \lambda \mathbf{v} \qquad \text{or} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \text{or} \qquad \begin{array}{lll} av_1 & +bv_2 & = \lambda v_1 \\ cv_1 & +dv_2 & = \lambda v_2 \end{array}$$

Any scalar $\lambda$ and vector $\mathbf{v}$ that satisfies this relationship are called an **eigenvalue** and an **eigenvector** (respectively) of the matrix.

Rewriting this system, we find that we would like to solve the following system of equations for the scalar $\lambda$ and vector $\mathbf{v}$ so that:

$$\begin{array}{llll} (a-\lambda)\, v_1 & + & b\ v_2 & = 0 \\ c\ v_1 & + & (d-\lambda) v_2 & = 0 \end{array} \qquad \text{or} \qquad \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using Cramer's Rule, we would find that if $(a-\lambda)(d-\lambda)-bc \neq 0$, then the (unique) solution to this system is $v_1 = 0$ and $v_2 = 0$.

However, by definition, an eigenvector should not be zero. Therefore, we conclude that, to get a non-zero eigenvector, the determinant should be zero:

$$(a-\lambda)(d-\lambda) - bc = 0$$

Expanding this gives the **characteristic equation** for the matrix $A$ (which is also the characteristic equation for the second order DE we would get from 7.1):

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \quad \text{or } \lambda^2 - Tr(A)\lambda + \det(A) = 0$$

**Summary**

To find the eigenvalues of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we first find the eigenvalues by solving the characteristic equation:

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

For each eigenvalue, solve the system of equations for $\mathbf{v}$:

$$\begin{array}{llll} (a-\lambda)\, v_1 & + & b\ v_2 & = 0 \\ c\ v_1 & + & (d-\lambda) v_2 & = 0 \end{array} \qquad \text{or} \qquad \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

**Worked Examples (Also see the HW Solutions)**

1.
$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

The trace is $1-4=-3$ and the determinant is $-4+6=2$. The characteristic equation is:

$$\lambda^2 + 3\lambda + 2 = 0$$

which we solve for $\lambda$. In this case, the quadratic factors and the eigenvalues are:

$$\lambda = -2, \quad -1$$

Now we get the eigenvectors:

- For $\lambda = -2$, we have the system:

$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{or } 3v_1 - 2v_2 = 0$$

  Choose any non-zero representative of this set to be the eigenvector. For example, $v_1 = 2, v_2 = 3$. In this case,

$$\lambda = -2 \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- For $\lambda = -1$, we have:

$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{or } 2v_1 - 2v_2 = 0$$

  Therefore, one representative eigenvector would be $v_1 = 1, v_2 = 1$, and

$$\lambda = -1, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2. Eigenvalues and eigenvectors may also be complex. For example,

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

The trace is 2 and the determinant is 5. The characteristic equation is:

$$\lambda^2 - 2\lambda + 5 = 0 \qquad (\lambda - 1)^2 + 4 = 0$$

which we solve for $\lambda$. In this case, completing the square seems faster than the quadratic formula, and: $\lambda = 1 \pm 2i$. For $\lambda = 1 + 2i$, solve the system on the left to get:

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v_2 = iv_1 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

You should verify that for $\lambda = 1 - 2i$, $\mathbf{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.