Review Questions: Exam 3

Chapter 6, Sections 5.1-5.3

You will be given the table of Laplace transforms.

1. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

   (a)
   $$f(t) = \begin{cases} 3, & 0 \le t < 2 \\ 6-t, & t \ge 2 \end{cases}$$

   (b)
   $$f(t) = \begin{cases} e^{-t}, & 0 \le t < 5 \\ -1, & t \ge 5 \end{cases}$$

2. Check your answers to Problem 1 by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.

3. Determine the Laplace transform:

   (a) $t^2 e^{-9t}$

   (b) $e^{2t} - t^3 - \sin(5t)$

   (c) $u_5(t)(t-5)^4$

   (d) $e^{3t}\sin(4t)$

   (e) $e^t\delta(t-3)$

   (f) $t^2 u_4(t)$

4. Find the inverse Laplace transform:

   (a) $\dfrac{2s-1}{s^2-4s+6}$

   (b) $\dfrac{7}{(s+3)^3}$

   (c) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$

   (d) $\dfrac{3s-2}{2s^2-16s+10}$

   (e) $\left(e^{-2s} - e^{-3s}\right)\dfrac{1}{s^2+s-6}$

5. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

   (a) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$

   (b) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$

   (c) $y'' - 4y' + 4y = t^2 e^t$, $y(0) = 0$, $y'(0) = 0$

6. Solve the given initial value problems using Laplace transforms:

   (a) $2y'' + y' + 2y = \delta(t-5)$, zero initial conditions.

   (b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$

   (c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$

   (d) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$

   (e) $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi)$, $y(0) = y'(0) = 0$. Write your answer in piecewise form.

7. Short Answer:

   (a) $\int_0^\infty \sin(3t)\delta(t - \frac{\pi}{2})\, dt =$

   (b) If $y'' + 2y' + 3y = 0$ and $y(0) = 1$, $y'(0) = -1$, compute $y''(0)$, $y'''(0)$, and $y^{(4)}(0)$.

   (c) Using your previous result, give the Taylor expansion of the solution to the differential equation using at least 5 terms.

   (d) If $y'(t) = \delta(t - c)$, what is $y(t)$?

   (e) What is the expected radius of convergence for the series expansion of $f(x) = 1/(x^2 + 2x + 5)$ if the series is based at $x_0 = 1$?

   (f) Use Laplace transforms to solve for $F(s)$, if

   $$f(t) + 2\int_0^t \cos(t - x) f(x)\, dx = \mathrm{e}^{-t}$$

   (So only solve for the transform of $f(t)$, don't invert it back).

   (g) In order for the Laplace transform of f to exist, f must be?

   (h) Can we assume that the solution to: $y'' + p(x)y' + q(x)y = u_3(x)$ is a power series?

8. More on Laplace Transforms:

   (a) Your friend tells you that the solutions to the IVPs:

   $$y''+2y'+y = 0,\ \ y(0) = 0,\ y'(0) = 1 \qquad \text{and} \qquad y''+2y'+y = \delta(t)\ \ y(0) = 0,\ y'(0) = 0$$

   are exactly the same. Are they really? Explain.

   (b) Let $f(t) = t$ and $g(t) = u_2(t)$.

      i. Use the Convolution Theorem to compute $f * g$.

      ii. Verify your answer by directly computing the integral.

9. Find the recurrence relation between the coefficients for the power series solutions to the following:

   (a) $2y'' + xy' + 3y = 0$, $x_0 = 0$.

   (b) $(1 - x)y'' + xy' - y = 0$, $x_0 = 0$

   (c) $y'' - xy' - y = 0$, $x_0 = 1$

10. Find the first 5 terms of the power series solution to $\mathrm{e}^x y'' + xy = 0$ if $y(0) = 1$ and $y'(0) = -1$.

11. Find the radius of convergence for the following series:

   (a) $\sum_{n=1}^{\infty} \sqrt{n} x^n$

   (b) $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}} (x + 3)^n$

   (c) $\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n}$