Homework Solutions: 1.1, 1.2

1 Section 1.1:

1. Problems 1-6

   In these problems, we want to compare and contrast the direction fields for the given (autonomous) differential equations of the form $y' = ay + b$. Once this is done, we want to be able to predict the direction field for the more general case.

   - Problem 1: $y' = 3 - 2y$. We should see that all solutions tend towards the equilibrium: $3 - 2y = 0$, or $y = 3/2$.
   - Problem 2: $y' = 2y - 3$. The equilibrium solution is $y = 3/2$, and if we begin on that solution, $y(t) = 3/2$ for all time. If the solution to the DE starts above $y = 3/2$, then the solution will tend to positive infinity. If the solution starts below $y = 3/2$, the solution tends to negative infinity.
   - Problem 3: $y' = 3 + 2y$. In this case, the equilibrium changes to $y = -3/2$, and like Problem 2, all other solutions will tend towards either positive or negative infinity (predictable when the solution starts above or below $-3/2$, respectively).
   - Problem 4: $y' = -1 - 2y$. The equilibrium is $y = -1/2$. All solutions will tend towards the equilibrium as $t \to \infty$.
   - Problem 5: $y' = 1 + 2y$ The equilibrium is again $y = -1/2$, except now the solutions move away from the equilibrium, going to $\pm\infty$ as $t \to \infty$ (again, that depends on the initial condition being above or below equilibrium).
   - Problem 6: $y' = y + 2$. The equilibrium is $y = -2$, and solutions again diverge to $\pm\infty$ as $t \to \infty$.

2. Problem 7: If we want all solutions to tend towards $y = 3$, that will need to be the equilibrium. Furthermore, in the equation $y' = ay + b$, the value of a needs to be negative. There are lots of possibilities; here is one:

   $$y' = -y + 3$$

3. Problem 9: All solutions tend away from $y = 2$. In this case, the value of a in $y' = ay+b$ needs to be positive, and we can write something like:

   $$y' = y - 2$$

   Summary for Problems 1-9: For $y' = ay + b$, the equilibrium solution is where $y' = 0$, or where $ay + b = 0$. This gives:

   $$y = -b/a$$

We can tell if the equilibrium is *attracting* (all solutions tend towards the equilibrium) or *repelling* (all solutions tend away from equilibrium) based on the sign of a. If $a > 0$, the equilibrium is repelling. If $a < 0$, the equilibrium is attracting.

4. For Problems 26-32, use direction fields (with sample solutions) in Maple. Here I will summarize what you should see, here are the basic Maple commands that I used:

```
with(DEtools):
DE26:=diff(y(t),t)=-2+t-y(t);
DE30:=diff(y(t),t)=3*sin(t)+1+y(t);
DE32:=diff(y(t),t)=-(2*t+y(t))/(2*y(t));
DEplot(DE26,y(t),t=-3..6,y=-4..4,[[y(0)=0.5],[y(0)=-0.5],[y(0)=-2],[y(0)=-4]]);
dsolve(DE26,y(t));
DEplot(DE30,y(t),t=-3..12,y=-5..5,[[y(0)=0],[y(0)=-1],[y(0)=1],[y(0)=-5/2]]);
dsolve(DE30,y(t));
DEplot(DE32,y(t),t=-2..2,y=-2..2);
```

- Problem 26: In problem 26, we see that all solutions tend towards the line $y(t) = -3 + t$ as $t \to \infty$. This result does not depend on the initial condition (I got this line by looking at Maple's solution to the DE).
- Problem 30: In this problem, if the initial condition is greater than $y = -5/2$, then the solution tends towards positive infinity as $t \to \infty$. If the initial condition is less than $y = -5/2$, then the solution moves to $-\infty$ as $t \to \infty$. If the initial condition is $y(0) = -5/2$, then the solution stays bounded- It stays on the curve
$$y(t) = -\frac{3}{2}\cos(t) - \frac{3}{2}\sin(t) - 1$$
for all time. (I got this function by looking at Maple's solution)
- Problem 32: This is where graphical analysis can play a large role in understanding the solutions to the differential equation. You should try to have Maple give you a solution using `dsolve`, but it won't be terribly useful.

  Here we see that all solutions starting above the $t-$axis will rotate (counterclockwise) around to the positive $t-$axis, and solutions that start below will rotate clockwise towards the positive $t-$axis. Thus, it appears that all solutions are converging to $y(t) = 0$.

2 Section 1.2:

1. Problem 1(a,b). Use Maple to get the pictures:

```
with(DEtools):
DE01a:=diff(y(t),t)=-y(t)+5;
DE01b:=diff(y(t),t)=-2*y(t)+5;
dfieldplot(DE01a,y(t),t=-3..3,y=-2..8);
dfieldplot(DE01b,y(t),t=-3..3,y=-4..4);
```

2. Problem 3: $y' = -ay + b$

   (a) The solution is found by:

   $$y' = -a\left(y - \frac{b}{a}\right) \quad \Rightarrow \quad \frac{1}{y - b/a}\,dy = -a\,dt \quad \Rightarrow \quad \int \frac{1}{y - b/a}\,dy = \int -a\,dt \Rightarrow$$

   $$\ln|y - b/a| = -at + C \quad \Rightarrow \quad y - \frac{b}{a} = \mathrm{e}^{-at+C} = \mathrm{e}^{-at}\mathrm{e}^{C} = A\mathrm{e}^{-at}$$

   So that the solution is:

   $$y(t) = \frac{b}{a} + A\mathrm{e}^{-at}$$

   (b) Your graph in this case should have a horizontal solution (the equilibrium solution) at $y = b/a$. The slopes above the equilibrium should go down, the slope below should point up.

   (c) Describe how the solution changes under each of the following conditions:

      i. a increases: This makes the solutions go to equilibrium faster than before (the slopes are made more steep). Changing a and leaving b fixed also makes the equilibrium get smaller.

      ii. b increases: Does not change the rate at which the solutions go to the equilibrium, but does change the equilibrium (if b increases, the equilibrium also increases).

      iii. Both a, b increase, but the ratio b/a stays fixed. This will change the rate at which solutions go to the equilibrium, which stays fixed.

3. Problem 5: Undetermined Coefficients.

   In this problem, we want to compare the solutions to:

   $$y' = ay \qquad \text{versus} \qquad y' = ay - b$$

   The solution to the first equation is: $y(t) = A\mathrm{e}^{at}$. To find the solution to the second, we **assume** that the solution is of the form:

   $$y(t) = A\mathrm{e}^{at} + k$$

   for some unknown k. Our problem is now to find k, which we do by substituting our guess into the differential equation.

   The left hand side of the D.E. is just y', so if $y = A\mathrm{e}^{at} + k$, then $y' = aA\mathrm{e}^{at}$.

   The right hand side of the D.E. is $ay - b$, so if $y = A\mathrm{e}^{at} + k$, this becomes $a(A\mathrm{e}^{at} + k) - b$.

   Now equate the left and right hand sides, and solve for k:

   $$aA\mathrm{e}^{at} = aA\mathrm{e}^{at} + ak - b \quad \Rightarrow \quad 0 = ak - b \quad \Rightarrow \quad k = b/a$$

Therefore, the overall solution is (what we had before):

$$y(t) = A\mathrm{e}^{at} + \frac{b}{a}$$

4. Problem 6: Solve $y' = -ay + b$ using the previous technique.

   We start with $y' = -ay$. The solution to this is $y(t) = A\mathrm{e}^{-at}$. Next we assume the solution to $y' = -ay + b$ is of the form:

   $$y(t) = A\mathrm{e}^{-at} + k$$

   for some unknown constant k. Substitute our guess into the differential equation. The left- and right- hand sides are:

   $$y' = -aA\mathrm{e}^{-at} \qquad -ay + b = -a\left(A\mathrm{e}^{-at} + k\right) + b$$

   Setting these equal and solving for k:

   $$-aA\mathrm{e}^{-at} = -aA\mathrm{e}^{-at} - ak + b \quad 0 = -ak + b \quad k = b/a$$

   so that:

   $$y(t) = A\mathrm{e}^{-at} + \frac{b}{a}$$

5. Problem 7 (Field Mice): $p' = \frac{1}{2}p - 450$

   (a) From the previous two problems (or with the technique from the Chapter), we can write down the solution:

   $$\frac{dp}{dt} = \frac{1}{2}(p - 900) \qquad \frac{dp}{p - 900} = \frac{1}{2}\,dt$$

   And integrate both sides:

   $$\ln|p - 900| = \frac{1}{2}t + C \quad \Rightarrow \quad p(t) = A\mathrm{e}^{(1/2)t} + 900$$

   Now, if $p(0) = 850$, we can get the particular solution (solve for A):

   $$p(0) = A + 900 = 850 \quad \Rightarrow \quad A = -50$$

   Therefore, $p(t) = -50\mathrm{e}^{(1/2)t} + 900$. To say that the population became extinct means that the population is zero. Set $p(t) = 0$ and solve for t:

   $$-50\mathrm{e}^{(1/2)t} + 900 = 0 \quad \Rightarrow \quad \mathrm{e}^{(1/2)t} = 18 \quad \Rightarrow t = 2\ln(18) \approx 5.78$$

(b) Similarly, if $p(0) = p_0$, with $0 < p_0 < 900$,

$$p_0 = A + 900 \Rightarrow A = p_0 - 900$$

and:

$$(p_0 - 900)e^{(1/2)t} + 900 = 0 \quad \Rightarrow \quad e^{(1/2)t} = \frac{-900}{p_0 - 900} = \frac{900}{900 - p_0}$$

(I wrote the last fraction like that so it would be clear that this is a positive number before we take the log of both sides)
Therefore, our conclusion is: Given $p' = \frac{1}{2}p - 450$, $p(0) = p_0$, where $0 < p_0 < 900$, then the time at which extinction occurs is:

$$t = 2\ln\left(\frac{900}{900 - p_0}\right)$$

(c) Find the initial population if the population becomes extinct in one year. Note that t is measured in months, so that would mean that we want to solve our general equation for p_0 if $p(12) = 0$. We can use our last result:

$$12 = 2\ln\left(\frac{900}{900 - p_0}\right)$$

Solve for p_0:

$$\frac{900}{900 - p_0} = e^6 \quad \Rightarrow \quad 900e^{-6} = 900 - p_0 \quad \Rightarrow \quad p_0 = 900 - 900e^{-6}$$

6. Problem 15 (Newton's Law of Cooling):
We are given:

$$\frac{du}{dt} = -k(u - T), \qquad u(0) = u_0$$

We can solve this either directly or using the techniques from this HW. Directly,

$$\frac{1}{u - T}\,dt = -k\,dt \quad \Rightarrow \quad \int \frac{1}{u - T}\,du = \int -k\,dt \quad \Rightarrow \quad \ln|u - T| = -kt + C$$

Now solve for $u(t)$:

$$u - T = e^{-kt+c} = e^{-kt}e^{c} = Ae^{-kt}$$

Also, find A in terms of the initial condition, $u(0) = u_0$:

$$u(0) = A + T = u + 0 \quad \Rightarrow \quad A = u_0 - T$$

In conclusion, the temperature at any time t:

$$u(t) = (u_0 - T)e^{-kt} + T$$

Part (b) is a little trickier, in that we need to properly translate the statement:

> Let τ be the time at which the initial temperature difference, $u_0 - T$ has been reduced by half. Find the relation between k and τ

If $u(t)$ is the actual temperature at time t, then $u(t) - T$ is the temperature *difference* at any time t between $u(t)$ and T. The statement is then translated to read:

$$u(\tau) - T = \frac{1}{2}(u_0 - T)$$

Now substitute and solve for k:

$$(u_0 - T)\mathrm{e}^{-k\tau} + T - T = \frac{1}{2}(u_0 - T)$$

So that:

$$\mathrm{e}^{-k\tau} = \frac{1}{2} \quad \Rightarrow \quad -k\tau = \ln(1/2) = -\ln(2) \quad \Rightarrow \quad k = \ln(2)/\tau$$