Section 1.3

1. Problem 1: Order is 2, and it is linear (divide by the leading t^2)
2. Problem 3: Order is 4, and it is linear.
3. Problem 5: Order is 2, and nonlinear (because of $\sin(t+y)$ term).
4. Problem 7: Do you know the definition of $\cosh(t)$? See our class website before doing this problem- There are practice problems there). You might use the definition directly, or from the practice sheet, see that:

   $$\frac{d}{dx}(\cosh(x)) = \sinh(x) \qquad \frac{d}{dx}(\sinh(x)) = \cosh(x)$$

   Now, to solve problem 7, we want to verify that either $y(t) = \mathrm{e}^t$ or $y(t) = \cosh(t)$ satisfies the differential equation: $y'' - y = 0$.

   If $y(t) = \mathrm{e}^t$, then $y'(t) = \mathrm{e}^t$, and $y''(t) = \mathrm{e}^t$, so

   $$y'' - y = \mathrm{e}^t - \mathrm{e}^t = 0$$

   If $y(t) = \cosh(t)$, then $y' = \sinh(t)$ and $y''(t) = \cosh(t)$, so again,

   $$y'' - y = \cosh(t) - \cosh(t) = 0$$

5. Problem 9: Show that $y(t) = 3t + t^2$ satisfies the ODE: $ty' - y = t^2$.

   First compute the derivative, then substitute into the expression:

   $$y' = 3 + 2t$$

   so that:

   $$ty' - y = t(3+2t) - (3t + t^2) = 3t + 2t^2 - 3t - t^2 = t^2$$

6. Problem 14: Show that the function

   $$y(t) = \mathrm{e}^{t^2} \int_0^t \mathrm{e}^{-s^2}\, ds + \mathrm{e}^{t^2}$$

   solves: $y' - 2ty = 1$.

   To show this directly, we need to recall how to differentiate a function like:

   $$g(t) = \int_0^t f(s)\, ds$$

   From the Fundamental Theorem of Calculus, $g'(t) = f(t)$.

Therefore, if $y(t)$ is as given above, the derivative is found by using the product rule:

$$y' = \left(2te^{t^2}\right) \cdot \int_0^t e^{-s^2}\, ds + e^{t^2} e^{-t^2} + 2te^{t^2}$$

If we simplify a bit, and subtract:

$$y' = 2te^{t^2} \int_0^t e^{-s^2}\, ds + 1 + 2te^{t^2}$$

$$-2ty = -2t\left(e^{t^2} \int_0^t e^{-s^2}\, ds + e^{t^2}\right)$$

We see that the only remaining term is 1.

(NOTE: In Section 2.1, we'll see where this strange integral is coming from)

7. Problem 15: We did something similar in class: If $y = e^{rt}$, substitute it into the differential equation-

$$y' + 2y = 0 \quad \Rightarrow \quad re^{rt} + 2e^{rt} = 0$$

Now solve for r:

$$(r+2)e^{rt} = 0 \quad \Rightarrow \quad r + 2 = 0 \quad \Rightarrow \quad r = -2$$

Note that $e^{rt} = 0$ has no solution.

Conclusion: $y(t) = e^{-2t}$.

Side Remark: We solved this in Section 1.2 by doing this:

$$y' = -2y \quad \Rightarrow \quad \frac{1}{y}\, dy = -2\, dt \quad \Rightarrow \quad \int \frac{1}{y}\, dy = -2 \int dt$$

so that:

$$\ln|y| = -2t + c \quad \Rightarrow \quad y(t) = Ae^{-2t}$$

8. Same setup as Problem 15: If $y(t) = e^{rt}$,

$$y'(t) = re^{rt} \qquad y''(t) = r^2 e^{rt}$$

Substitute these into the DE: $y'' - y' - 6y = 0$ and solve for r:

$$r^2 e^{rt} + re^{rt} - 6e^{rt} = 0 \quad \Rightarrow \quad e^{rt}\left(r^2 + r - 6\right) = 0$$

Again, $e^{rt} = 0$ has no solution, so just solve:

$$r^2 + r - 6 = 0 \qquad (r+3)(r-2) = 0 \qquad r = -3, 2$$

Either $y = e^{-3t}$ or $y = e^{2t}$ will solve the DE.

9. Problem 19: In this case, assume $y = t^r$, so $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substitute these into the DE:

$$t^2 y'' + 4ty' + 2y = 0 \quad \Rightarrow \quad t^2 \cdot r(r-1)t^{r-2} + 4t \cdot rt^{r-1} + 2t^r = 0$$

Simplify and factor out t^r:

$$t^r (r(r-1) + 4r + 2) = 0$$

This equation must be true for ALL $t > 0$ (given in the problem), so $t^r = 0$ does not give a solution. Solve for r:

$$r^2 - r + 4r + 2 = 0 \quad \Rightarrow \quad r^2 + 3r + 2 = 0 \quad \Rightarrow \quad (r+1)(r+2) = 0$$

Therefore, $y(t) = \frac{1}{t}$ and $y(t) = \frac{1}{t^2}$ solve the differential equation.

10. Problem 21: The order is 2, linear.

11. Problem 25: Show that each of these:

$$u(x,y) = \cos(x)\cosh(y) \qquad u(x,y) = \ln(x^2 + y^2)$$

solve the Partial Differential Equation (PDE):

$$u_{xx} + u_{yy} = 0$$

If $u(x,y) = \cos(x)\cosh(y)$, then

$$u_x = -\sin(x)\cosh(y) \qquad u_{xx} = -\cos(x)\cosh(y)$$

Similarly,

$$u_y = \cos(x)\sinh(y) \qquad u_{yy} = \cos(x)\cosh(y)$$

And if we add u_{xx} to u_{yy}, we get zero.

If $u(x,y) = \ln(x^2 + y^2)$, then:

$$u_x = \frac{2x}{x^2 + y^2} \qquad u_{xx} = \frac{-2(x^2 - y^2)}{(x^2 + y^2)^2}$$

Similarly,

$$u_y = \frac{2y}{x^2 + y^2} \qquad u_{yy} = \frac{-2(y^2 - x^2)}{(x^2 + y^2)^2}$$

And again we see that if we add u_{xx} and u_{yy}, we get zero.