Solutions: Section 2.1

1. Problem 1: See the Maple worksheet to get the direction field. You should see that it looks like all solutions are approaching some curve (maybe a line?) as $t \to \infty$. To be more analytic, let us solve the DE using the Method of Integrating Factors.

$$y' + 3y = t + \mathrm{e}^{-2t} \quad \Rightarrow \quad \mathrm{e}^{3t}\left(y' + 3y\right) = \mathrm{e}^{3t}\left(t + \mathrm{e}^{-2t}\right) \quad \Rightarrow \quad \left(\mathrm{e}^{3t}y(t)\right)' = t\mathrm{e}^{3t} + \mathrm{e}^{t}$$

Integrate both sides *Hint:* We need to use "integration by parts" to integrate $t\mathrm{e}^{3t}$. Using a table as in class:

$+$	t	e^{3t}
$-$	1	$(1/3)\mathrm{e}^{3t}$
$+$	0	$(1/9)\mathrm{e}^{3t}$

$$\Rightarrow \quad \int t\mathrm{e}^{3t}\,dt = \frac{1}{3}\mathrm{e}^{3t} - \frac{1}{9}\mathrm{e}^{3t}$$

Putting it all together,

$$\mathrm{e}^{3t}y(t) = \frac{1}{3}t\mathrm{e}^{3t} - \frac{1}{9}\mathrm{e}^{3t} + \mathrm{e}^{t} + C$$

so that

$$y(t) = \frac{1}{3}t - \frac{1}{9} + \frac{1}{\mathrm{e}^{-2t}} + \frac{C}{\mathrm{e}^{3t}}$$

Notice that the last two terms go to zero as $t \to \infty$, so we see that $y(t)$ does approach a line:

$$\frac{1}{3}t - \frac{1}{9}$$

as $t \to \infty$.

2. Problem 3: See Maple for the direction field. Very similar situation to Problem 1. Let's go ahead and solve:

$$y' + y = t\mathrm{e}^{-t} + 1$$

Multiply both sides by $\mathrm{e}^{\int p(t)\,dt} = \mathrm{e}^{t}$:

$$\mathrm{e}^{t}\left(y' + y\right) = t + \mathrm{e}^{t} \quad \Rightarrow \quad \left(\mathrm{e}^{t}y(t)\right)' = t + \mathrm{e}^{t}$$

Integrate both sides:

$$\mathrm{e}^{t}y(t) = \frac{1}{2}t^2 + \mathrm{e}^{t} + C \quad \Rightarrow \quad y(t) = \frac{1}{2}t^2\mathrm{e}^{-t} + 1 + C\mathrm{e}^{-t}$$

This could be written as:

$$y(t) = 1 + \frac{t^2}{2\mathrm{e}^{t}} + \frac{C}{\mathrm{e}^{t}}$$

so that it is clear that, as $t \to \infty$, $y(t) \to 1$, which we also see in the direction field.

3. Problem 11: See Maple for the direction field, where it looks like all solutions are approaching some periodic function as $t \to \infty$. Let's solve it:

$$y' + y = 5\sin(2t)$$

As in the last exercise, multiply both sides by e^t:

$$\mathrm{e}^t(y' + y) = 5\mathrm{e}^t \sin(2t) \quad \Rightarrow \quad \left(\mathrm{e}^t y(t)\right)' = 5\mathrm{e}^t \sin(2t)$$

To integrate the right-hand-side of this equation, we will need to use integration by parts twice. In tabular form:

$+$	e^t	$\sin(2t)$
$-$	e^t	$-(1/2)\cos(2t)$
$+$	e^t	$-(1/4)\sin(2t)$

$$\Rightarrow \quad \int \mathrm{e}^t \sin(2t)\, dt = -\frac{1}{2}\mathrm{e}^t \cos(2t) + \frac{1}{4}\mathrm{e}^t \sin(2t) - \frac{1}{4}\int \mathrm{e}^t \sin(2t)\, dt$$

Add the last integral to the left:

$$\frac{5}{4}\int \mathrm{e}^t \sin(2t)\, dt = -\frac{1}{2}\mathrm{e}^t \cos(2t) + \frac{1}{4}\mathrm{e}^t \sin(2t)$$

so that:

$$\int \mathrm{e}^t \sin(2t)\, dt = -\frac{2}{5}\mathrm{e}^t \cos(2t) + \frac{1}{5}\mathrm{e}^t \sin(2t) + C_1$$

Going back to the differential equation,

$$\mathrm{e}^t y(t) = -2\mathrm{e}^t \cos(2t) + \mathrm{e}^t \sin(2t) + C_2$$

so that the general solution is:

$$y(t) = -2\cos(2t) + \sin(2t) + C_2 \mathrm{e}^{-t}$$

We see that, as $t \to \infty$, $y(t)$ does indeed go to a periodic function.

In problems 13, 15, 16, solve the IVP. For these problems, I will leave the details out, but I will give the integrating factor. Be sure to ask in class if you're not sure how to solve them!

4. Problem 13: (You'll need to integrate by parts!)

$$y' - y = 2t\mathrm{e}^{2t} \qquad \mathrm{e}^{\int p(t)\, dt} = \mathrm{e}^{-t}$$

$$y(t) = \mathrm{e}^{2t}(2t - 2) + 3\mathrm{e}^t$$

5. Problem 15:

$$ty' + 2y = t^2 - t + 1$$

Be sure to put in standard form before solving:

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t} \qquad \mathrm{e}^{\int p(t)\,dt} = t^2$$

and

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

6. Problem 16: In this problem, the integrating factor is again t^2:

$$y' + \frac{2}{t} \cdot y = \frac{\cos(t)}{t^2} \qquad \Rightarrow \qquad y(t) = \frac{\sin(t)}{t^2}$$

7. Problem 21: See the example Maple worksheet to get the direction field. To solve the IVP (with $y(0) = a$):

$$y' = -\frac{1}{2}y = 2\cos(t)$$

The integrating factor is: $\mathrm{e}^{-(1/2)t}$:

$$\left(\mathrm{e}^{-(1/2)t}y\right)' = 2\mathrm{e}^{-(1/2)t}\cos(t)$$

Use integration by parts twice:

$+$	$\cos(t)$	$\mathrm{e}^{-(1/2)t}$
$-$	$-\sin(t)$	$-2\mathrm{e}^{-(1/2)t}$
$+$	$-\cos(t)$	$4\mathrm{e}^{-(1/2)t}$

$\Rightarrow$

$$\int \mathrm{e}^{-(1/2)t}\cos(t)\,dt = -2\mathrm{e}^{-(1/2)t}\cos(t) + 4\mathrm{e}^{-(1/2)t}\sin(t) - 4\int \mathrm{e}^{-(1/2)t}\cos(t)\,dt$$

Add the last integral to both sides and divide by 5:

$$\int \mathrm{e}^{-(1/2)t}\cos(t)\,dt = -\frac{2}{5}\mathrm{e}^{-(1/2)t}\cos(t) + \frac{4}{5}\mathrm{e}^{-(1/2)t}\sin(t) + C$$

Going back to get the solution (be sure to multiply the antiderivative by 2:

$$\mathrm{e}^{-(1/2)t}y = -\frac{4}{5}\mathrm{e}^{-(1/2)t}\cos(t) + \frac{8}{5}\mathrm{e}^{-(1/2)t}\sin(t) + C$$

So that:

$$y(t) = -\frac{4}{5}\cos(t) + \frac{8}{5}\sin(t) + C\mathrm{e}^{(1/2)t}$$

Solve for the constant in terms of the initial condition $y(0) = a$:

$$a = -\frac{4}{5} + C \quad \Rightarrow \quad C = a + \frac{4}{5}$$

The solution to the IVP is:

$$y(t) = -\frac{4}{5}\cos(t) + \frac{8}{5}\sin(t) + \left(a + \frac{4}{5}\right) e^{(1/2)t}$$

In particular, we see that if $y(0) = a = -4/5$, then the solution will be the periodic part (and will not become unbounded). Otherwise (because of the exponential function), all other solutions will become unbounded as $t \to \infty$.

8. Problem 24: See the Maple sample for the direction field.

   To solve the IVP, first write in standard form, then find the integrating factor:

$$y' + \frac{t+1}{t} y = 2e^{-t}, \qquad t > 0, \quad y(1) = a$$

The integrating factor: First compute the antiderivative-

$$\int \frac{t+1}{t}\, dt = \int 1 + \frac{1}{t}\, dt = t + \ln(t), \qquad t > 0$$

And exponentiate:

$$e^{\int p(t)\, dt} = e^{t + \ln(t)} = e^{t} e^{\ln(t)} = te^{t}$$

Now,

$$\left(te^{t} y(t)\right)' = 2t \quad \Rightarrow \quad te^{t} y(t) = t^2 + C$$

so that the general solution is:

$$y(t) = \frac{t^2 + C}{te^{t}}$$

Solve in terms of a:

$$y(1) = \frac{1 + C}{e} = a \quad \Rightarrow \quad C = ae - 1$$

so that:

$$y(t) = \frac{t^2 + (ae - 1)}{te^{t}}$$

Analysis: If the constant $ae - 1 = 0$, then $y(t)$ becomes te^{-t}, which is zero at time $t = 0$. Otherwise, all other solutions are not defined at time $t = 0$. The value of a is then $a = 1/e \approx 0.3679$. Furthermore, as $t \to 0$, the solution will tend to zero (as does all solutions).

9. Problem 27: Solve the IVP

$$y' + \frac{1}{2}y = 2\cos(t), \qquad y(0) = -1$$

Using the integrating factor of $e^{(1/2)t}$,

$$\left(e^{(1/2)t}y(t)\right)' = 2e^{(1/2)t}\cos(t)$$

To integrate the right hand side of the equation, use integration by parts twice (since we've showed this a couple of times, I leave it out here):

$$e^{(1/2)t}y(t) = \frac{4}{5}e^{(1/2)t}\cos(t) + \frac{8}{5}e^{(1/2)t}\sin(t) + C$$

so that:

$$y(t) = \frac{4}{5}\cos(t) + \frac{8}{5}\sin(t) + Ce^{-(1/2)t}$$

Solve for C:

$$-1 = \frac{4}{5} + C \quad \Rightarrow \quad C = -\frac{9}{5}$$

and the solution to the IVP is:

$$y(t) = \frac{4}{5}\cos(t) + \frac{8}{5}\sin(t) - \frac{9}{5}e^{-(1/2)t}$$

We now want to find the coordinates of the first local maximum, $t > 0$. This means that we want to solve for the first t for which the derivative is zero. Unfortunately, we cannot do this exactly, so we can use Maple to find a numerical approximation. Here is the Maple code to do this:

```
DE27:=diff(y(t),t)+(1/2)*y(t)=2*cos(t);
Y27:=dsolve({DE27,y(0)=-1},y(t));
dy:=diff(rhs(Y27),t);
plot(dy,t=0..3);
tsol:=fsolve(dy=0,t=0..2);
evalf(subs(t=tsol,rhs(Y27)));
```

so the coordinates are approximately $(1.3643, 0.8201)$.

Notes about the Maple commands:

- If you look at `Y27`, you'll see that:

  ```
  Y27 := y(t) = 4/5*cos(t)+8/5*sin(t)-9/5*exp(-1/2*t)
  ```

  Therefore, to plot the function, we need the *right hand side* of `Y27`. In Maple, this is `rhs(Y27)`.

- To get the FIRST value of t, I need a rough estimate for the `fsolve` function. That's why we plot the derivative first. You see in the `fsolve` line, `t=0..2`, which is the estimate I got from the graph.

10. Problem 29: Solve the IVP:

$$y' + \frac{1}{4}y = 3 + 2\cos(2t) \qquad y(0) = 0$$

To find the solution, we see that the integrating factor is $\mathrm{e}^{(1/4)t}$. Multiply both sides by the I.F. and integrate. Note that again we'll need to integrate by parts twice to evaluate:

$$\int \mathrm{e}^{(1/4)t}\cos(2t)\,dt = \frac{4}{65}\mathrm{e}^{(1/4)t}\cos(2t) + \frac{32}{65}\mathrm{e}^{(1/4)t}\sin(2t)$$

Therefore,

$$\left(\mathrm{e}^{(1/4)t}y(t)\right) = 12\mathrm{e}^{(1/4)t} + \frac{8}{65}\mathrm{e}^{(1/4)t}\cos(2t) + \frac{64}{65}\mathrm{e}^{(1/4)t}\sin(2t) + C$$

so that:

$$y(t) = 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) + C\mathrm{e}^{-(1/4)t}$$

Solve for C:

$$0 = 12 + \frac{8}{65} + C \quad \Rightarrow \quad C = -\frac{788}{65}$$

so that the overall solution is:

$$y(t) = 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) - \frac{788}{65}\mathrm{e}^{-(1/4)t}$$

As $t \to \infty$, the last term (with the exponential) drops out, leaving the rest. That means the solution will become periodic (oscillating about the line $y = 12$) as $t \to \infty$.

To solve for the first value of t for which the function crosses the line $y = 12$, we need to solve the following equation for the first t for which:

$$12 = 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) - \frac{788}{65}\mathrm{e}^{-(1/4)t}$$

Or, the first time that:

$$\frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) - \frac{788}{65}\mathrm{e}^{-(1/4)t} = 0$$

We cannot solve this analytically, so we look for a numerical approximation in Maple:

```
DE29:=diff(y(t),t)+(1/4)*y(t)=3+2*cos(2*t);
Y29:=dsolve({DE29,y(0)=0},y(t));
plot(rhs(Y29)-12,t=9..10.2);
fsolve(rhs(Y29)-12=0,t=9.8..10.2);
```

and $t \approx 10.0658$.

11. Problem 30: No Maple here! Solve the IVP:

$$y' - y = 1 + 3\sin(t) \qquad y(0) = y_0$$

This is very similar to Problem 29. Note that:

$$\int \mathrm{e}^{-t}\sin(t)\,dt = -\frac{1}{2}\mathrm{e}^{-t}\left(\cos(t) + \sin(t)\right)$$

Therefore, the general solution is (details left out):

$$y(t) = -1 - \frac{3}{2}\left(\cos(t) + \sin(t)\right) + \left(\frac{5}{2} + y_0\right)\mathrm{e}^{t}$$

To keep the solution finite (or bounded) as $t \to \infty$, we must find y_0 so that the exponential term drops out- This means that $y_0 = -5/2$.

12. Problem 32: Show that all solutions to:

$$2y' + ty = 2$$

approach a finite limit as $t \to \infty$, and find the limiting value.

We'll find the general solution by getting the Integrating Factor:

$$p(t) = \frac{1}{2}t \quad \Rightarrow \quad \mathrm{e}^{\int p(t)\,dt} = \mathrm{e}^{(1/4)t^2}$$

Now,

$$\left(\mathrm{e}^{(1/4)t^2}y(t)\right)' = \mathrm{e}^{(1/4)t^2}$$

so that

$$\mathrm{e}^{(1/4)t^2}y(t) = \int_0^t \mathrm{e}^{(1/4)x^2}\,dx + C$$

I'm writing this antiderivative as a *particular* antiderivative so that (i) the constant of integration comes out and (ii) it is clear how to differentiate the integral. Using this notation,

$$y(t) = \mathrm{e}^{-(1/4)t^2}\int_0^t \mathrm{e}^{(1/4)x^2}\,dx + C\mathrm{e}^{-(1/4)t^2} = \frac{\int_0^t \mathrm{e}^{(1/4)x^2}\,dx + C}{\mathrm{e}^{(1/4)t^2}}$$

We set it up this way since the book hints that we should try to use L'Hospital's rule. A quick check of the numerator and denominator should convince you that it is appropriate here (that is, we have "∞/∞").

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{\mathrm{e}^{(1/4)t^2}}{(1/2)t\mathrm{e}^{(1/4)t^2}} = \lim_{t\to\infty}\frac{2}{t} = 0$$

13. For Problem 34, we are asked to go backwards: Given a desired solution, construct an appropriate first order linear differential equation for it.

    For this problem, we want $y(t) \to 3$ as $t \to \infty$. Here are two possible ways of proceeding:

    - Suppose $y(t) = 3 + C\mathrm{e}^{-t}$ (so it looks a lot like the solutions we got for the previous HW problems). Then $y' = -C\mathrm{e}^{-t}$, and we see that:

    $$y' + y = 3$$

      (I'll leave the verification to you).

    - As another possible approach, we could take:

    $$y(t) = 3 + \frac{C}{t^2}$$

      Now, $y' = -2C/t^3$. We see that if we take ty' and add it to $2y$, the terms with C cancel and we're left with 6. Therefore, the ODE is:

    $$ty' + 2y = 6$$

      (I'll leave the verification to you).

14. Problem 35 is similar. There are many ways of constructing such a differential equation- It's easiest to start with a desired solution. We'll again show two possibilities:

    - If we would like $y(t) = 3 - t + C\mathrm{e}^{-3t}$, then $y' = -1 - 3C\mathrm{e}^{-3t}$, and:

    $$y' + 3y = 8 - 3t$$

    - If we would like $y(t) = 3 - t + \frac{C}{t}$, then $y' = -1 - C/t^2$, and we see that:

    $$ty' + y = 3 - 2t$$

15. Problems 38, 39: Let's wait until Section 3.7 for this method.