Solutions: Section 2.2

1. Problem 1: Give the general solution: $y' = x^2/y$

$$y\,dy = x^2\,dx \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{1}{3}x^3 + C$$

2. Problem 3: Give the general solution to $y' + y^2\sin(x) = 0$.

   First write in standard form:

$$\frac{dy}{dx} = -y^2\sin(x) \quad \Rightarrow \quad -\frac{1}{y^2}\,dy = \sin(x)\,dx$$

   Before going any further, notice that we have divided by y, so we need to say that this is value as long as $y(x) \neq 0$. In fact, we see that the function $y(x) = 0$ IS a possible solution.

   With that restriction in mind, we proceed by integrating both sides to get:

$$\frac{1}{y} = -\cos(x) + C \quad \Rightarrow \quad y = \frac{1}{C - \cos(x)}$$

3. Problem 7: Give the general solution:

$$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$$

   First, note that dy/dx exists as long as $y \neq e^y$. With that requirement, we can proceed:

$$(y + e^y)\ dy = \left(x + e^{-x}\right)\ dx$$

   Integrating, we get:

$$\frac{1}{2}y^2 + e^y = \frac{1}{2}x^2 - e^{-x} + C$$

   In this case, we cannot algebraically isolate y, so we'll leave our answer in this form (we could multiply by two).

4. Problem 9: Let $y' = (1 - 2x)y^2, \qquad y(0) = -1/6$.

   First, we find the solution. Before we divide by y, we should make the note that $y \neq 0$. We also see that $y(x) = 0$ is a possible solution (although NOT a solution that satisfies the initial condition).

   Now solve:

$$\int y^{-2}\,dy = \int (1 - 2x)\,dx \quad \Rightarrow \quad -y^{-1} = x - x^2 + C$$

   Solve for the initial value:

$$6 = 0 + C \Rightarrow C = 6$$

The solution is (solve for y):

$$y(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)}$$

The solution is valid only on $-2 < x < 3$, and we could plot this by hand (also see the Maple worksheet).

5. Problem 11: $x\,dx + y\mathrm{e}^{-x} dy = 0, \qquad y(0) = 1$

   To solve, first get into a standard form, multiplying by e^x, and integrate (integration by parts for the right hand side):

$$\int y\,dy = -\int x\mathrm{e}^x\,dx \quad \Rightarrow \quad \frac{1}{2}y^2 = -x\mathrm{e}^x + \mathrm{e}^x + C$$

   We could solve for the constant before isolating y:

$$\frac{1}{2} = 0 + 1 + C \quad C = -\frac{1}{2}$$

   Now solve for y:

$$y^2 = 2\mathrm{e}^x(x-1) - \frac{1}{2}$$

   and take the positive root, since $y(0) = +1$.

$$y = \sqrt{2\mathrm{e}^x(1-x) - 1}$$

   The solution exists as long as:

$$2\mathrm{e}^x(1-x) - 1 \geq 0$$

   We use Maple to solve where this is equal to zero (see the Worksheet online). From that, we see that $-1.678 \leq x \leq 0.768$

6. Problem 14:

$$\frac{dy}{dx} = xy^3(1+x^2)^{-1/2} \qquad y(0) = 1$$

   Since we'll divide by y, we look at the case where $y = 0$. We see that it is a possible solution, but not for this initial value, therefore, $y \neq 0$:

$$\int y^{-3}\,dy = \int \frac{x}{\sqrt{x^2+1}}\,dx$$

   To integrate the right side of the equation, let $u = x^2 + 1$. Integrating, we get:

$$-\frac{1}{2}y^{-2} = \sqrt{x^2+1} + C \qquad \Rightarrow \qquad \frac{1}{y^2} = C_2 - 2\sqrt{x^2+1}$$

We could solve for the constant now: $1 = C_2 - 2$, so $C = 3$. Solve for y:

$$y(x) = \frac{1}{\sqrt{3 - \sqrt{x^2+1}}}$$

where we take the positive root since the initial condition was positive.

The solution will exist as long as the denominator is not zero. Solving,

$$3 - 2\sqrt{x^2+1} = 0 \qquad \sqrt{x^2+1} = 3/2 \qquad x = \pm\sqrt{5}/2$$

The solution is valid for $-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$. See Maple for the plot.

7. Problem 16:
$$\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3} \qquad y(0) = -\frac{1}{\sqrt{2}}$$
First, we notice that $y \neq 0$. Now separate the variables and integrate:
$$y^4 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$$
This might be a good time to solve for C: $C = 1/4$, so:
$$y^4 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{1}{4}$$
The right side of the equation seems to be a nice form. Try some algebra to simplify it:
$$\frac{1}{4}\left(x^4 + 2x^2 + 1\right) = \frac{1}{4}(x^2+1)^2$$
Now we can write the solution:
$$y^4 = \frac{1}{4}(x^2+1)^2 \quad \Rightarrow \quad y = -\frac{1}{\sqrt{2}}\sqrt{x^2+1}$$
This solution exists for all x (it is the bottom half of a hyperbola- see the Maple plot).

8. Problem 20: $y^2\sqrt{1-x^2}dy = \sin^1(x)\,dx$ with $y(0) = 1$.

To put into standard form, we'll be dividing so that $x \neq \pm 1$. In that case,
$$\int y^2\,dy = \int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$$
The right side of the equation is all set up for a u, du substitution, with $u = \sin^{-1}(x)$, $du = 1/\sqrt{x^2-1}\,dx$:
$$\frac{1}{3}y^3 = \frac{1}{2}(\arcsin(x))^2 + C$$

Solve for C, $\frac{1}{3} = 0 + C$ so that:

$$\frac{1}{3}y^3 = \frac{1}{2}\arcsin^2(x) + \frac{1}{3}$$

Now,

$$y(x) = \sqrt[3]{\frac{3}{2}\arcsin^2(x) + 1}$$

The domain of the inverse sine is: $-1 \le x \le 1$. However, we needed to exclude the endpoints. Therefore, the domain is:

$$-1 < x < 1$$

9. Problem 21: I'll start this off in standard form with a note that says that $y \neq 0, y \neq 2$. With these restrictions,

$$\int (3y^2 - 6y)\,dy = (1 + 3x^2)\,dx \quad \Rightarrow \quad y^3 - 3y^2 = x + x^3 + C$$

Solve for C using the initial condition, $y(0) = 1$: $C = -2$, and:

$$y^3 - 3y^2 = x + x^3 - 2$$

This is a solution in implicit form. We have vertical tangent lines where $y = 0$ and $y = 2$, so we can find the corresponding x values:

$$0 = x^3 + x - 2$$

By inspection, $x = 1$ (See Maple to get the full set of solutions). If $y = 2$, then $-4 = x^3 + x - 2$, or $0 = x^3 + x + 2$, and by inspection, $x = -1$.

Therefore, the solution exists for $-1 < x < 1$ (See the Maple plot for verification).

10. Problem 25: From what is given, we assume that $3 + 2y \neq 0$, and:

$$y' = \frac{2\cos(2x)}{3 + 2y} \quad \Rightarrow \quad (3 + 2y)\,dy = 2\cos(2x)\,dx$$

Integrate both sides, and use the initial condition $y(0) = -1$

$$3y + y^2 = \sin(2x) + C \quad \Rightarrow \quad -3 + 1 = 0 + C \quad \Rightarrow \quad C = -2$$

The implicit solution is:

$$y^2 + 3y = \sin(2x) - 2$$

We can solve this for y by completing the square:

$$y^2 + 3y = \left(y^2 + 3y + \frac{9}{4}\right) - \frac{9}{4} = \left(y + \frac{3}{2}\right)^2 - \frac{9}{4}$$

so that:

$$\left(y + \frac{3}{2}\right)^2 = \sin(2x) + \frac{1}{4} \Rightarrow y = -\frac{3}{2} + \sqrt{\sin(2x) + \frac{1}{4}}$$

(the positive root was chosen to match the initial condition).

11. Problem 27: First consider the solutions to the ODE,

$$y' = \frac{ty(4-y)}{3}$$

We see that $y(t) = 0$ and $y(t) = 4$ are possible solutions. Otherwise, we can divide by $y(4-y)$, and get:

$$\frac{1}{y(4-y)}\,dy = \frac{1}{3}t\,dt$$

Integrate the left side using partial fraction decomposition:

$$\frac{1}{y(4-y)} = \frac{1}{4}\cdot\frac{1}{y} + \frac{1}{4}\cdot\frac{1}{4-y}$$

Multiply by 4, and integrate:

$$\ln|y| - \ln|4-y| = \frac{2}{3}t^2 + C \qquad \Rightarrow \qquad \ln\left|\frac{y}{4-y}\right| = \frac{2}{3}t^2 + C$$

Therefore,

$$\frac{y}{4-y} = A\mathrm{e}^{(2/3)t^2} \qquad \text{and} \qquad \frac{y_0}{4-y_0} = A$$

Solve for y, where A is shown above:

$$y(t) = \frac{4A\mathrm{e}^{(2/3)t^2}}{1 + A\mathrm{e}^{(2/3)t^2}}$$

For the dependence of the solution on y_0, look at the direction field in Maple. We should see that $y(t) = 0$ and $y(t) = 4$ are indeed solutions. Furthermore, if $y_0 < 0$, $y(t) \to -\infty$ as $t \to \infty$. If $y_0 = 0$, $y(t) = 0$ for all time. If $0 < y_0 < 4$, $y(t) \to 4$ as $t \to \infty$. If $y_0 = 4$, $y(t) = 4$ for all time. Finally, if $y_0 > 4$, we see that $y(t) \to 4$ as $t \to \infty$.

NOTE: I would accept the above solution for $y(t)$ on a quiz or exam, however, it is better to simplify it a bit by dividing numerator and denominator by $A\mathrm{e}^{(2/3)t^2}$.