Solutions: Section 2.5

NOTE: Some of the graphs will be put online as images rather than PDF text.

1. Problem 1: Given $\frac{dy}{dt} = ay + by^2 = y(a + by)$ with $a, b > 0$. For the more general case, we will let y_0 be any real number.

   Always look for the equilibria first! In this case,

   $$y(a + by) = 0 \quad \Rightarrow \quad y = 0 \text{ or } y = -b/a$$

   To make the phase plot (graph of y' versus y), we note that $ay + by^2$ is a parabola opening upwards, and it intersects the $y-$axis at the equilibria, $y = 0$ and $y = -b/a$. From this graph, we see that $y = 0$ is an unstable equilibrium, and $y = -b/a$ is stable.

2. Problem 3: Given $\frac{dy}{dt} = y(y-1)(y-2)$, and let y_0 be any real number (the more general case).

   Then the phase plot is a cubic function going through the equilibria at $y = 0$, $y = 1$, $y = 2$. See the graph online.

3. Problem 7: With the DE,
   $$\frac{dy}{dt} = k(1 - y)^2$$
   the only equilibrium solution is: $k(1 - y)^2 = 0 \Rightarrow y = 1$. Graphing this as y' versus y, we get an upward parabola whose vertex is lying on the $y-$axis at $y = 1$.

   For part (b), see the graph online.

   For part (c), the DE is separable:

   $$\int \frac{1}{(1 - y)^2}\, dy = \int k\, dt \quad \Rightarrow \quad \frac{1}{1 - y} = kt + C$$

   (Use u, du substitution for the integral on the left side of the equation). At this stage, we might as well solve for the arbitrary constant:

   $$\frac{1}{1 - y_0} = 0 + C$$

   This is valid as long as $y_0 \neq 1$. In the case that $y_0 = 1$, the solution is $y(t) = 1$ (the equilibrium solution).

   Solving for y,

   $$1 - y = \frac{1}{kt + C} \quad \Rightarrow \quad y = 1 - \frac{1}{kt + \frac{1}{1-y_0}}$$

   Let us analyze this last equation: If $\frac{1}{1-y_0} > 0$, then as $t \to \infty$, $kt + \frac{1}{1-y_0} \to \infty$, so $y(t) \to 1$. Therefore, if $y_0 < 1$, $y(t) \to 1$ as $t \to \infty$ (as expected from the phase plot and direction field).

On the other hand, consider the case when $y_0 > 1$ (the case when $y_0 = 1$ gave an equilibrium solution). In this case, $\frac{1}{1-y_0}$ is negative, which means that there will be a vertical asymptote in positive time,

$$t = -\frac{1}{k(1-y_0)}$$

From our phase plot, we expect solutions with $y_0 > 1$ to go to $+\infty$- Does that occur algebraically?

$$y(t) = 1 - \frac{1}{kt + \frac{1}{1-y_0}} = 1 - \frac{\frac{1}{k}}{t + \frac{1}{k(1-y_0)}}$$

so we see that the denominator is approaching zero *from the left*, so that $y(t) \to +\infty$ as $t \to -1/(k(1-y_0))$ from the left.

4. Problem 8, 10, 11: These are all done graphically- See the attachments online.

5. Problem 22: Please be sure to read the description carefully- Nice intro to epidemiology.

   (a) The equilibria are at $y = 0$ and $y = 1$. The phase plot of $y' = \alpha y(1-y)$ is a parabola opening downward. A sketch of the phase plot shows that $y = 0$ is unstable and $y = 1$ is stable.

   (b) To solve this, we'll need to use partial fraction decomposition:

   $$\frac{1}{y(1-y)} dy = \alpha\, dt \Rightarrow \int \frac{1}{y} + \frac{1}{1-y}\, dy = \alpha t + C \Rightarrow \ln|y| - \ln|1-y| = \alpha t + C$$

   so that

   $$\ln\left|\frac{y}{1-y}\right| = \alpha t + C \quad \Rightarrow \quad \frac{y}{1-y} = A\mathrm{e}^{\alpha t}$$

   Solving for A, $y_0/(1-y_0) = A$. Keep this in mind, and let's solve for y first:

   $$y(t) = \frac{A\mathrm{e}^{\alpha t}}{1 + A\mathrm{e}^{\alpha t}}$$

   We will want to analyze what happens as $t \to \infty$, so it will be more convenient to divide numerator and denominator by $A\mathrm{e}^{\alpha t}$:

   $$y(t) = \frac{1}{\frac{1}{A}\mathrm{e}^{-\alpha t} + 1} = \frac{1}{\frac{1-y_0}{y_0}\mathrm{e}^{-\alpha t} + 1}$$

   This solution is valid as long as $y_0 \neq 0$ and $y_0 \neq 1$. In those cases, our solutions are the equilibrium solutions, $y(t) = 0$ and $y(t) = 1$. Now let us analyze the behavior of $y(t)$.

   We see that, as $t \to \infty$, $y(t) \to 1$. But this is not the end of the story: If a solution begins with $y_0 < 0$, for example, we know that the solution CANNOT approach 1 as $t \to \infty$, because that would mean it would have to cross $y(t) = 0$ (and solutions cannot intersect by the E& U Theorem).

The following is a much more detailed analysis than what was expected in the homework problem- However, read through it to see exactly what the behavior of all solutions looks like.

The only point that makes us pause is the denominator. Set it to zero and solve:

$$\frac{1-y_0}{y_0}\mathrm{e}^{-\alpha t} = -1 \quad \Rightarrow \quad \mathrm{e}^{-\alpha t} = \frac{y_0}{y_0 - 1} \quad \Rightarrow \quad t = -\frac{1}{\alpha} \cdot \ln\left(\frac{y_0}{y_0-1}\right)$$

Alternatively,

$$t = \frac{1}{\alpha} \cdot \ln\left(\frac{y_0-1}{y_0}\right) = \frac{1}{\alpha} \cdot \ln\left(1 - \frac{1}{y_0}\right)$$

The reason this is a nice way of analyzing t:

- If $y_0 > 1$, then we will be taking the log of a number less than 1 (which gives a negative value). In this case, t is negative and our solution $y(t)$ is valid for all $t > (1/\alpha)\ln(1 - (1/y_0))$, and $y(t) \to 1$ as $t \to \infty$.
- If $0 < y_0 < 1$, this denominator is never zero (no solution for t in the real numbers). In this case, $y(t)$ is valid for ALL t (not just positive), and again the limit as $t \to \infty$ is 1.
- If $y_0 < 0$, then the solution is valid for:

  $$-\infty < t < \frac{1}{\alpha}\ln\left(1 - \frac{1}{y_0}\right)$$

  so that $y(t)$ has a vertical asymptote on the positive t axis. In this case, it is not appropriate to take the limit as $t \to \infty$.

6. Problem 23:

   First solve $y' = -\beta y$, which is $y(t) = y_0\mathrm{e}^{-\beta t}$.

   NOTE: There is a misprint in Problem 23, in defining dx/dt. The disease *spreads* (or INCREASES) at a rate proportional to the number of carrier-susceptible interactions ($x-$ and $y-$ interactions), which means that the constant in front should be POSITIVE.

   We are told to substitute this into the DE:

   $$\frac{dx}{dt} = +\alpha xy = \alpha x\left(y_0\mathrm{e}^{-\beta t}\right)$$

Solve this separable equation for $x(t)$:

$$\int \frac{1}{x}\,dx = \alpha y_0 \int \mathrm{e}^{-\beta t}\,dt \quad \Rightarrow \quad \ln|x| = \frac{-\alpha \cdot y_0}{\beta}\mathrm{e}^{-\beta t} + C$$

Solving for the initial value,

$$C = \ln|x_0| + \frac{\alpha \cdot y_0}{\beta}$$

so that:

$$\ln|x| = \frac{\alpha \cdot y_0}{\beta}\left(1 - \mathrm{e}^{-\beta t}\right) + \ln|x_0|$$

Finally, exponentiating both sides:

$$x(t) = x_0 \mathrm{e}^{\frac{\alpha \cdot y_0}{\beta}\left(1 - \mathrm{e}^{-\beta t}\right)}$$

And the limit as $t \to \infty$ of $x(t)$ is $x_0 \mathrm{e}^{\frac{\alpha \cdot y_0}{\beta}}$

7. Problem 25: The basic idea behind problems 25 and 26 is that there is a new parameter, a. By changing this parameter, we can change the *number* and *type* of the equilibrium solutions.

   In Problem 25, the equilibrium solutions are given by:

   $$\frac{dy}{dt} = 0 \Rightarrow a - y^2 = 0 \quad \Rightarrow \quad y = \pm\sqrt{a}$$

   Graphically in the phase plot, $y' = -y^2$ is an upside down parabola, and $-y^2 + a$ simply translates the parabola up and down.

   Therefore, in words:

   - If $a < 0$, we have no equilibrium solutions.
   - If $a = 0$, we have a single equilibrium solution at $a = 0$, and it is *semistable*. Since y' is always negative (and zero at $y = 0$), in the direction field, solutions that begin above $y_0 = 0$ decrease to zero, and solutions that begin below $y_0 = 0$ decrease to negative infinity.
   - If $a > 0$, we have two equilibrium solutions (at $\sqrt{a}$ and $-\sqrt{a}$). The positive root is a *stable* equilibrium, and the negative root is an *unstable* equilibrium.

   We can summarize this graphically in Figure 2.5.10 on page 93.

8. Problem 26: Finding the equilibrium:

   $$y(a - y^2) = 0$$

   We see that $y(t) = 0$ is ALWAYS an equilibrium solution for any value of a. The other solutions will be the same as before (we'll have to re-do our stability analysis):

- If $a < 0$, the only equilibrium is $y(t) = 0$, and this is stable.
- If $a = 0$, same situation.
- If $a > 0$, two new equilibria appear, $y(t) = \pm\sqrt{a}$. Now, $y(t) = 0$ switches stability (it is now unstable), and the two new equilibria, $y(t) = \pm\sqrt{a}$ are both stable.