

Solutions: Section 2.6

The material beginning on page 98, Integrating Factors for Exact Equations, will not be on the exam.

1. Problem 1: $(2x + 3) + (2y - 2)y' = 0$

We want $f_x = M(x, y) = 2x + 3$ and $f_y = N(x, y) = 2y - 2$. We check if this is possible:

$$M_y = 0 \quad N_x = 0$$

Now antidifferentiate M with respect to x :

$$f(x, y) = \int M(x, y) dx = \int 2x + 3 dx = x^2 + 3x + g(y)$$

where g is some unknown function of y . Two ways of proceeding (which are equivalent). I'll list both methods for this problem:

- Try to get f from N , and compare:

$$f(x, y) = \int N(x, y) dy = \int 2y - 2 dy = y^2 - 2y + \hat{g}(x)$$

where \hat{g} is an unknown function of x . Comparing this to what we had before, we see that:

$$f(x, y) = x^2 + 3x + y^2 - 2y$$

so the implicit solution is: $x^2 + 3x + y^2 - 2y = C$

NOTE: You can always check your answer!

- Another method: Starting from where we left off,

$$f(x, y) = x^2 + 3x + g(y)$$

we can see what g needs to be in order for $f_y = N$, or:

$$f_y = g'(y) = 2y - 2 = N$$

In that case, $g(y) = y^2 - 2y$, and $f(x, y) = x^2 + 3x + y^2 - 2y$.

The implicit solution is: $x^2 + 3x + y^2 - 2y = C$

2. Problem 3: $(3x^2 - 2xy + 2) dx + (6y^2 - x^2 + 3) dy = 0$

Check to see if the equation is exact:

$$M_y = -2x \quad N_x = -2x$$

So yes. Now we'll antidifferentiate M with respect to x :

$$f(x, y) = \int M dx = \int 3x^2 - 2xy + 2 dx = x^3 - x^2y + 2x + g(y)$$

Check to see if f_y is equal to N :

$$f_y = -x^2 + g'(y) = 6y^2 - x^2 + 3$$

so that $g'(y) = 6y^2 + 3$. That gives $g(y) = 2y^3 + 3y$. Put this back in to get the full solution, $f(x, y) = c$:

$$x^3 - x^2y + 2x + 2y^3 + 3y = C$$

3. Problem 4: $(2xy^2 + 2y) + (2x^2y + 2x)\frac{dy}{dx} = 0$

Check for “exactness”:

$$M_y = 4xy + 2 \quad N_x = 4xy + 2$$

Now set:

$$f(x, y) = \int M \, dx = \int 2xy^2 + 2y \, dx = x^2y^2 + 2xy + g(y)$$

And check to see that $f_y = N$:

$$f_y = 2x^2y + 2x + g'(y) = 2x^2y + 2x$$

In this case, $g'(y) = 0$, and we don't need to add $g(y)$. The implicit solution:

$$x^2y^2 + 2xy = C$$

4. Problem 13: $(2x - y) \, dx + (2y - x) \, dy = 0$

Check first: $M_y = -1 = N_x$, so the DE is exact.

Now,

$$f(x, y) = \int M \, dx = \int 2x - y \, dx = x^2 - xy + g(y)$$

where we check f_y to make it equal to $N(x, y)$:

$$f_y = -x + g'(y) = 2y - x \quad \Rightarrow \quad g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2$$

Our implicit solution is:

$$x^2 - xy + y^2 = C$$

With the initial condition $y(1) = 3$, we get:

$$1^2 - (1)(3) + 3^2 = C \quad \Rightarrow \quad C = 7$$

The solution to the IVP is:

$$x^2 - xy + y^2 = 7$$

Now we can isolate y and find the restrictions on x (think quadratic formula in y , parentheses added for emphasis):

$$y^2 + (-x)y + (x^2 - 7) = 0$$

Now,

$$y = \frac{x \pm \sqrt{x^2 - 4(x^2 - 7)}}{2} = \frac{x \pm \sqrt{28 - 3x^2}}{2}$$

Take the positive root since $y(1) = 3$.

The restriction on x would be that $28 - 3x^2 \geq 0$. Therefore,

$$-\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}}$$

5. Problem 15: $(xy^2 + bx^2y) dx + (x + y)x^2 dy = 0$

First, for this to be exact:

$$M_y = 2xy + bx^2 = 3x^2 + 2xy = N_x$$

So $b = 3$. With this, find the solution to the DE:

$$f(x, y) = \int M dx = \int xy^2 + 3x^2y dx = \frac{1}{2}x^2y^2 + x^3y + g(y)$$

And solve for $g(y)$:

$$f_y = x^2y + x^3 + g'(y) = x^3 + x^2y$$

So we didn't need $g(y)$. This leaves:

$$\frac{1}{2}x^2y^2 + x^3y = C$$

6. Problem 18: Done in Class. The idea is:

$$M(x) + N(y) \frac{dy}{dx} = 0$$

Given that, $M_y = 0$ and $N_x = 0$, so the equation is exact.

We would solve:

$$f(x, y) = \int M(x) dx + g(y)$$

Taking f_y and equating it to N : $f_y = g'(y) = N(y)$. Therefore, $g(y) = \int N(y) dy$, and the solution is:

$$f(x, y) = \int M(x) dx + \int N(y) dy = C$$

which is what we do with separable equations.

7. Problem 19: We see that:

$$x^2y^3 + (x + xy^2)\frac{dy}{dx} = 0$$

is not separable, since $(x^2y^3)_y = 3x^2y^2$, but $(x + xy^2)_x = 1 + y^2$. However, after multiplication by $\mu(x, y) = 1/(xy^3)$, we get:

$$\frac{x^2y^3}{xy^3} + \frac{x(1 + y^2)}{xy^3}\frac{dy}{dx} = 0$$

Simplify:

$$x + (y^{-3} + y^{-1})\frac{dy}{dx} = 0$$

Note that this becomes *separable*, so it is also exact.

Using the methods from this section,

$$f(x, y) = \int M dx = \frac{1}{2}x^2 + g(y)$$

and $f_y = g'(y) = N = y^{-3} + y^{-1}$. Therefore,

$$g(y) = \int y^{-3} + \frac{1}{y} dy = -\frac{1}{2}y^{-2} + \ln(y)$$

The solution is:

$$\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln(y) = C$$

8. Problem 22:

In this case, when we multiply by the given integrating factor:

$$xe^x(x + 2)\sin(y) + xe^xx\cos(y) = 0$$

Expand it to make the partial derivatives a bit easier:

$$x^2e^x\sin(y) + 2xe^x\sin(y) + (x^2e^x\cos(y))\frac{dy}{dx} = 0$$

Now check the partials:

$$M_y = x^2e^x\cos(y) + 2xe^x\cos(y)$$

and

$$N_x = \cos(y)(2xe^x + x^2e^x)$$

And these are the same (so it is exact). In fact, this equation is also separable (this is done for fun):

$$\frac{dy}{dx} = -\frac{\sin(y)(x^2 + 2x)e^x}{\cos(y)x^2e^x} = -\tan(y)\left(1 + \frac{2}{x}\right) \Rightarrow \int \cot(y) dy = -\int 1 + \frac{2}{x} dx$$

Let's see if we can avoid¹ the antiderivative of $\cot(y)$:

$$f(x, y) = \int M \, dx = \sin(y) \int e^x (x^2 + 2x) \, dx + g(y) = x^2 e^x \sin(y) + g(y)$$

Now check $f_y = N$:

$$f_y = x^2 e^x \cos(y) + g'(y) = x^2 e^x \cos(y)$$

so we did not need $g(y)$ in this case. The solution is:

$$x^2 e^x \sin(y) = C$$

¹Actually, this is not bad- Do it with a u, du substitution.