Selected Solutions: 3.2

1. Problem 16, 3.2: We are told that $y = \sin(t^2)$ is a solution- Substitute it into the DE to determine $p(t)$ and $q(t)$:

$$y = \sin(t^2) \qquad y' = 2t\cos(t^2) \qquad y'' = 2\cos(t^2) - 4t^2\sin(t^2)$$

   Substituting into $y'' + p(t)y' + q(t)y = 0$, we get:

$$\begin{array}{rlr} y'' & = 2\cos(t^2) & -4t^2\sin(t^2) \\ p(t)y' & = p(t)2t\cos(t^2) & \\ q(t)y & = & q(t)\sin(t^2) \\ \hline 0 & = (2 + 2tp(t))\cos(t^2) & +(q(t) - 4t^2)\sin(t^2) \end{array}$$

   Therefore, if $\sin(t^2)$ was a solution, $p(t) = \frac{-1}{t}$ and $q(t) = 4t^2$. However, this would make $p(t)$ not continuous at $t = 0$.

2. Problem 18, Sect 3.2:

   We're given the Wronskian and a function $f(t) = t$. Find the function g:

$$W(f, g)(t) = \begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} = tg'(t) - g(t) = t^2\mathrm{e}^t$$

   That is a linear differential equation in g:

$$g'(t) - \frac{1}{t}g(t) = t\mathrm{e}^t$$

   The integrating factor: $\mathrm{e}^{-\int (1/t)\,dt} = \frac{1}{t}$, so:

$$\left(\frac{g(t)}{t}\right)' = \mathrm{e}^t \quad \Rightarrow \quad g(t)/t = \mathrm{e}^t + C \quad \Rightarrow \quad g(t) = t(\mathrm{e}^t + C)$$

3. Problem 22, Section 3.2: Find the fundamental set (Theorem 3.2.5) if $y'' + 4y' + 3y = 0$.

   We know the general solution is $C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-3t}$. In Theorem 3.2.5, we construct two solutions that are guaranteed to form a fundamental set:

   - y_1 solves the ODE with initial conditions $y(1) = 1, y'(1) = 0$:

$$\begin{array}{rl} C_1\mathrm{e}^{-1} + C_2\mathrm{e}^{-3} & = 1 \\ -C_1\mathrm{e}^{-1} - 3C_2\mathrm{e}^{-3} & = 0 \end{array} \Rightarrow C_1 = \frac{3}{2}\mathrm{e}, C_2 = -\frac{1}{2}\mathrm{e}^3$$

     Therefore, $y_1(t) = \frac{3\mathrm{e}}{2}\mathrm{e}^{-t} - \frac{\mathrm{e}^3}{2}\mathrm{e}^{-3t}$.

- Similarly, y_2 solves the ODE with I.C.s: $y(1) = 0, y'(1) = 1$:

$$\begin{array}{rl} C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-3} & = 0 \\ -C_1 \mathrm{e}^{-1} - 3C_2 \mathrm{e}^{-3} & = 1 \end{array} \Rightarrow C_1 = \frac{\mathrm{e}}{2}, C_2 = -\frac{\mathrm{e}^3}{2}$$

  Therefore, $y_2(t) = \frac{\mathrm{e}}{2}\mathrm{e}^{-t} - \frac{\mathrm{e}^3}{2}\mathrm{e}^{-3t}$

Even though y_1 and y_2 look a lot alike, Theorem 3.2.5 guarantees that they are linearly independent, and that they form a fundamental set.

Note that Theorem 3.2.5 is more of a formal result than something we would actually compute with- However, it does give conditions on which we can always guarantee that we can find a fundamental set of solutions.

4. Problem 26, Section 3.2: The verification is straightforward.

   Before we consider the question of whether we have a fundamental set, look at where the solutions would be valid.

   We will have a discontinuity of p, q, g at the $x-$values where:

$$1 - x\cot(x) = 0 \quad \Rightarrow \quad x\cot(x) = 1 \quad \Rightarrow \quad x\frac{\cos(x)}{\sin(x)} = 1 \quad \Rightarrow \quad x\cos(x) = \sin(x)$$

   Therefore, existence and uniqueness is only guaranteed on intervals which avoid these points.

   Going to the original question, does x and $\sin(x)$ constitute a fundamental set? Look at the Wronskian:

$$W(x, \sin(x)) = \begin{vmatrix} x & \sin(x) \\ 1 & \cos(x) \end{vmatrix} = x\cos(x) - \sin(x)$$

   This looks very familiar! The Wronskian will be non-zero for intervals which also satisfy the existence and uniqueness theorem.

   Finally, is the given interval one such example? The expression will not be zero on $0 < x < \pi$ (check this graphically).

5. Problem 27, Section 3.2: Just a couple of notes here. You should find that y_1, y_3 do form a fundamental set; y_2, y_3 do NOT form a fundamental set.

   To show that y_1, y_4 do form a fundamental set, notice that, since y_1, y_2 do form a fundamental set,

$$y_1 y_2' - y_1' y_2 \neq 0 \text{ at } t_0$$

   Now form the Wronskian between y_1 and y_4:

$$W(y_1, y_4) = \begin{vmatrix} y_1 & y_1 + 2y_2 \\ y_1' & y_1' + 2y_2' \end{vmatrix} = y_1 y_1' + 2y_1 y_2' - y_1 y_1' - 2y_1' y_2 = 2W(y_1, y_2) \neq 0$$

   The last set, y_4, y_5 does NOT form a fundamental set. You can show that $y_4 = y_1 + 2y_2$, and $y_5 = 2y_4$.