Selected Solutions: 3.3

Some notes before we get started

- The relationship between linear independence and the Wronskian:

  From the definition of linear independence, we are looking at solving the following equation for k_1 and k_2:
  $$k_1 f(t) + k_2 g(t) = 0$$
  We see immediately that $k_1 = k_2 = 0$ is ALWAYS one possible solution. The big question is whether or not there are other, non-zero solutions.

  If f, g are differentiable, then we can create an additional system of equations:
  $$k_1 f'(t) + k_2 g'(t) = 0$$
  From Cramer's Rule, if $W(f, g)(t_0) \neq 0$, then the ONLY solution to our system is $k_1 = k_2 = 0$, and we are finished (Conclusion: f, g are linearly independent).

  The problem is what happens if the Wronskian is zero? We said that in this case, Cramer's Rule does not apply, and we have to go back to the original equations- go back to see if there are non-zero constants that solve the system.

  The homework problems show that, if f, g are two generic functions, then it is possible that the Wronskian is zero for all t, but f, g are linearly independent. It is also possible that f, g are linearly dependent- We have to check it out (typically, try evaluating $k_1 f(t) + k_2 g(t) = 0$ at two distinct time values).

  However, Abel's Theorem tells us that if f, g are two SOLUTIONS to: $y'' + p(t)y' + q(t)y = 0$, then the Wronskian is either ALWAYS zero (f, g are dependent), or NEVER zero (f, g are Independent) on the interval for which the solutions are valid.

- Here's a handy **Theorem:**

  Let f, g be non-zero functions. Then: $g(t)$ is a constant multiple of $f(t)$ *if and only if* f, g are linearly dependent.

  Proof: First, if $g(t) = cf(t)$, then $-cf(t) + g(t) = 0$ for all t. Therefore, we have non-zero constants $k_1 = -c$ and $k_2 = 1$ so that $k_1 f(t) + k_2 g(t) = 0$, valid for all t.

  On the other hand, if there are non-zero constants such that $k_1 f(t) + k_2 g(t) = 0$, then we can solve for f in terms of g:
  $$f(t) = -\frac{k_2}{k_1} g(t)$$

1. Problem 7: Let $y_1 = 3t$ and $y_2 = |t|$.

   We won't use the Wronskian since y_2 is not differentiable at $t = 0$. The question of independence is tied to the interval on which we are checking, and it is important in this example:

   - If the interval I is contained in $t > 0$, then $y_2 = t$. The functions y_1, y_2 are constant multiples of each other, and are therefore linearly dependent.
   - If the interval I is contained in $t < 0$, then $y_2 = -t$, and again y_1, y_2 are constant multiples of each other. Therefore, the are again linearly dependent.
   - If the (open) interval I contains the origin, we can always find a positive number, $\epsilon > 0$ and its negative, $-\epsilon$ within the interval I. Using these two points, we check the definition of linear independence:

     $$\begin{array}{rllll} \text{At } t = \epsilon: & k_1 t + k_2|t| & = k_1\epsilon + k_2\epsilon & = \epsilon(k_1 + k_2) & = 0 \\ \text{At } t = -\epsilon: & k_1 t + k_2|t| & = -k_1\epsilon + k_2\epsilon & = \epsilon(-k_1 + k_2) & = 0 \end{array}$$

     From this, $k_1 = -k_2$ and $k_1 = k_2$. These can both be true ONLY if $k_1 = k_2 = 0$.

   Conclusion: On any open interval containing the origin, t and $|t|$ are linearly independent. Otherwise, they are linearly dependent.

2. Problem 8: Exactly the same argument as Problem 7.

3. Problem 9: If $W(f, g) = t\sin^2(t)$, are the functions linearly dependent or independent?

   We see that the Wronskian is equal to zero only at isolated points of time ($t = 0, t = k\pi$). Therefore, any open interval I will contain points other than these, which implies that on any open interval I, we can find t_0 such that $W(f, g)(t_0) \neq 0$. Using Theorem 3.3.1, this means that f, g are linearly INDEPENDENT.

4. Problem 10: Same reasoning as Problem 9.

5. Problem 11:

   If y_1, y_2 are linearly independent solutions of the given ODE, then:

   $$W(y_1, y_2) = C\mathrm{e}^{-\int p(x)\,dx} \neq 0$$

   Can we express $W(c_1y_1, c_2y_2)$ in terms of $W(y_1, y_2)$?

   $$W(c_1y_1, c_2y_2) = \begin{vmatrix} c_1y_1 & c_2y_2 \\ c_1y_1' & c_2y_2' \end{vmatrix} = c_1c_2(y_1y_2' - y_2y_1') = c_1c_2W(y_1, y_2) \neq 0$$

6. Problem 12: You might see the nice relationship between $W(y_1, y_2)$ and $W(y_1+y_2, y_1-y_2)$:

$$W(y_1+y_2, y_1-y_2) = \begin{vmatrix} y_1+y_2 & y_1-y_2 \\ y_1'+y_2' & y_1'-y_2' \end{vmatrix} = -2(y_1y_2' - y_2y_1') = -2W(y_1, y_2)$$

So either both Wronskians are zero or not zero.

Conclusion: y_1, y_2 are linearly independent if and only if y_1+y_2, y_1-y_2 are linearly independent.

7. Problem 13: An extension of Problem 12. Directly compute the Wronskian and do some algebra to show that:

$$W(a_1y_1+a_2y_2, b_1y_1+b_2y_2) = (a_1b_2 - a_2b_1)W(y_1, y_2)$$

Note to people who have had linear algebra:

Do you recognize this as: $\det(AB) = \det(A)\det(B)$?

8. Problem 15: Use Abel's Theorem, where

$$-\int p(t)\,dt = \int \frac{t(t+2)}{t^2}\,dt = \int \frac{t+2}{t}\,dt = \int 1 + \frac{2}{t}\,dt = t + 2\ln(t)$$

so that the Wronskian is $Ct^2\mathrm{e}^t$.

9. Problem 20: If y_1, y_2 are linearly independent solutions to $ty'' + 2y' + t\mathrm{e}^t y = 0$ and $W(y_1, y_2)(1) = 2$, compute $W(y_1, y_2)(5)$.

We compute the Wronskian:

$$W(y_1, y_2)(t) = \mathrm{e}^{-\int 2/t\,dt} = \frac{C}{t^2}$$

If $W(y_1, y_2)(1) = 2$, then $C = 2$, and

$$W(y_1, y_2)(5) = \frac{2}{5^2} = \frac{2}{25}$$

10. Problem 23: If f, g, h are differentiable, show that $W(fg, fh) = f^2W(g, h)$. First compare the starting Wronskian to what we want at the end:

$$W(fg, fh) = \begin{vmatrix} fg & fh \\ (fg)' & (fh)' \end{vmatrix} \qquad f^2W(g,h) = f^2\begin{vmatrix} g & h \\ g' & h' \end{vmatrix} = f^2(gh' - g'h)$$

The algebra is straightforward from here...

11. Problem 24: If there is a t_0 so that $y_1(t_0) = 0$ and $y_2(t_0) = 0$, then:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} 0 & 0 \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = 0$$

So, by Abel's Theorem they cannot be linearly independent on the given interval I (and so they do not form a fundamental set).

12. Problem 25: Same reasoning- There exists t_0 so that $y_1'(t_0) = 0$ and $y_2'(t_0) = 0$, so $W(y_1, y_2)(t_0) = 0$.

13. Problem 27: Show that t and t^2 are linearly independent on any interval. By definition, we are looking for constants k_1, k_2 so that:

$$k_1 t + k_2 t^2 = 0 \qquad t(k_1 + k_2 t) = 0$$

If $t \neq 0$, then $k_1 + k_2 t = 0$ for all time t. The only way this could happen is if $k_1 = k_2 = 0$. Therefore, t and t^2 are linearly independent (on any open interval- Note that an open interval could not contain only the point $t = 0$, there would always be some value not zero in I).

The Wronskian is:

$$W(t, t^2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2$$

We see that the Wronskian is zero at $t = 0$. That implies that, if t, t^2 are solutions to: $y'' + p(t)y' + q(t)y = 0$, then the interval on which that solution is valid cannot contain $t = 0$. In fact, we could solve directly for what p and q would have to be by substituting $y_1 = t$ and $y_2 = t^2$ into the differential equation:

$$y_1 = t, \quad y_1' = 1, \quad y_1'' = 0 \quad \Rightarrow \quad p(t) + q(t)t = 0 \quad \Rightarrow \quad p(t) = -tq(t)$$

Similarly,

$$y_2 = t^2, \quad y_2' = 2t, \quad y_2'' = 2 \quad \Rightarrow \quad 2 + p(t)(2t) + q(t)t^2 = 0$$

With these two equations, $q(t) = \frac{2}{t^2}$ and $p(t) = -\frac{2}{t}$, which are continuous for $t > 0$ or $t < 0$, which is what we anticipated.