Selected Problems from Section 3.4

1. Problem 25. Let $y'' + 2y' + 6y = 0$ with $y(0) = 2$ and $y'(0) = \alpha \geq 0$.

   (a) Solve it: The characteristic equation is $r^2 + 2r + 6 = 0$. Therefore, $r = -1 \pm \sqrt{5}\, i$ and the general solution is:

   $$y(t) = \mathrm{e}^{-t}\left(C_1 \cos(\sqrt{5}\, t) + C_2 \sin(\sqrt{5}\, t)\right)$$

   With the initial conditions, find that $C_1 = 2$ and $C_2 = \frac{\alpha+2}{\sqrt{5}}$, so that the solution to the IVP is:

   $$y(t) = \mathrm{e}^{-t}\left(2\cos(\sqrt{5}\, t) + \frac{\alpha+2}{\sqrt{5}}\sin(\sqrt{5}\, t)\right)$$

   (b) Find α so that $y(1) = 0$: Algebraically, we get:

   $$-2\sqrt{5}\,\frac{\cos(\sqrt{5})}{\sin(\sqrt{5})} - 2 = \alpha$$

   (c) Find the smallest value of $t > 0$ so that $y(t) = 0$. We'll write our answer as a function of α.
   Algebraically, solving this:

   $$0 = \mathrm{e}^{-t}\left(2\cos(\sqrt{5}\, t) + \frac{\alpha+2}{\sqrt{5}}\sin(\sqrt{5}\, t)\right)$$

   will get us to:

   $$\frac{-2\cos(\sqrt{5}\, t)}{\sin(\sqrt{5}\, t)} = \frac{\alpha+2}{\sqrt{5}}$$

   We want to solve this for t, remembering that we want the smallest $t > 0$. I think it is easiest to work with the tangent rather than the cotangent,

   $$\tan(\sqrt{5}\, t) = \frac{-2\sqrt{5}}{\alpha+2}$$

   Now consider the graph of $\tan(\sqrt{5}\, t)$. It looks much like tangent, except that it is periodic with period $\pi/\sqrt{5}$.
   If we consider the graph of $\tan(\sqrt{5}\, t)$, as shown in Maple in Figure 1, we see that solving directly for t:

   $$t = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{-2\sqrt{5}}{\alpha+2}\right)$$

   will put us one period ($\pi/\sqrt{5}$) short. Therefore, the full solution is:

   $$t = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{-2\sqrt{5}}{\alpha+2}\right) + \frac{\pi}{\sqrt{5}}$$

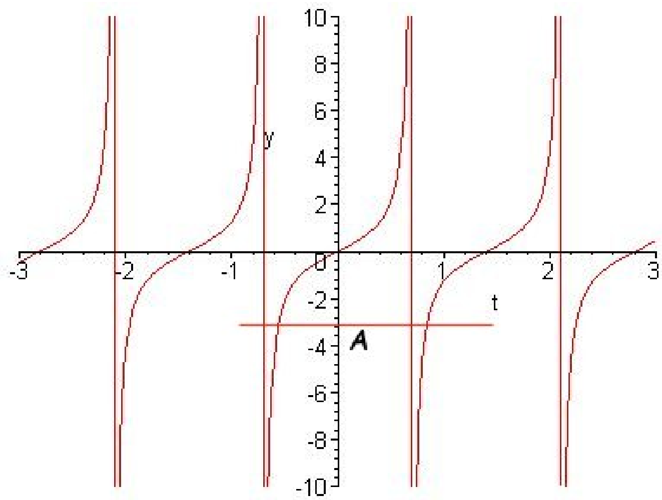

Figure 1: Plot of $tan(\sqrt{5}t)$ and a sample value of $-2\sqrt{5}/(\alpha+2)$ marked as A. Using the standard restriction (so that the tangent is invertible), we see that we are one period too short- Add one period to get the actual solution.

Taking the limit as $\alpha \to \infty$, we get that $t = \pi/\sqrt{5}$.
NOTE: The inverse tangent is an odd function, so one could write the solution as:

$$t = \frac{\pi}{\sqrt{5}} - \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{2\sqrt{5}}{\alpha+2}\right)$$

2. Problem 29:

   Using Euler's Formula and noting that

   $$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta)$$

   we can simplify the following:

   $$e^{it} + e^{-it} = \cos(t) + i\sin(t) + \cos(t) - i\sin(t) = 2\cos(t)$$

   and

   $$e^{it} - e^{-it} = \cos(t) + i\sin(t) - \cos(t) + i\sin(t) = 2i\sin(t)$$

3. Problem 32: Differentiate expressions in i as if i is a constant (it is a constant):

   If $\phi = u + iv$, then $\phi' = u' + iv'$ and $\phi'' = u'' + iv''$. Therefore, substituting this into the differential equation, we get:

   $$(u'' + iv'') + p(t)(u' + iv') + q(t)(u + iv) = 0$$

Separate into real and imaginary components:

$$(u'' + p(t)u' + q(t)u) + i(v'' + p(t)v' + q(t)v) = 0$$

We note that if $a + ib = 0$, then $a = 0$ and $b = 0$. Apply this principle to the above equation so that

$$u'' + p(t)u' + q(t)u = 0 \quad \text{and} \quad v'' + p(t)v' + q(t)v = 0$$

Therefore, $u(t)$ and $v(t)$ must separately solve the homogeneous second order linear differential equation.