Selected Problems from Section 3.6

1. For problems 19-22, be sure that you can write down the form of the particular solution. To find the solution, see the Maple file on our website.

   In the Maple worksheet, note how we write the 2[d] derivative, y'': `diff(y(t),t$2)`

2. Problem 28: Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin(m\pi t)$$

   where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, 2, \ldots, N$. (Note: This type of problem does arise from some engineering models!)

   Writing this out, we see:

$$y'' + \lambda^2 y = a_1 \sin(\pi t) + a_2 \sin(2\pi t) + a_2 \sin(3\pi t) + \ldots + a_N \sin(N\pi t)$$

   We solve this piece by piece; first get the homogeneous part of the solution:

$$y_h(t) = C_1 \cos(\lambda t) + C_2 \sin(\lambda t)$$

   Now the m^{th} function $g(t)$ is given by: $a_m \sin(m\pi t)$, so our m^{th} guess is:

$$y_p = A\cos(m\pi t) + B\sin(m\pi t)$$

   Substitute this into the differential equation, where

$$y_p'' = -Am^2\pi^2 \cos(m\pi t) - Bm^2\pi^2 \sin(m\pi t)$$

   so $y_p'' + \lambda^2 y_p$ is:

$$A(\lambda^2 - m^2\pi^2)\cos(m\pi t) + B(\lambda^2 - m^2\pi^2)\sin(m\pi t)$$

   which is where the text answer comes from.

3. Problem 31 continues from Problem 38 in Section 3.5.

   If Y_1 and Y_2 are solutions to:

$$ay'' + by' + cy = g(t)$$

   with $a, b, c > 0$, and if y_1, y_2 form a fundamental set to the homogeneous equation, then:

   - In problem 38, Sect 3.5, we said that $c_1 y_1 + c_2 y_2 \to 0$ as $t \to \infty$.
   - In this section, we said that $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$.

By the previous two items, $Y_1 - Y_2 \to 0$ as $t \to \infty$

If $b = 0$, then $c_1 y_1 + c_2 y_2$ do not go to zero, but oscillate. Therefore, $Y_1 - Y_2$ would oscillate (but remain bounded) as $t \to \infty$.

4. Problem 32: If $g(t) = d$, write the solution:

$$ay'' + by' + cy = d$$

The homogeneous solution is $C_1 y_1 + C_2 y_2$ (neither y_1 nor y_2 are constant if $a, b, c > 0$).

The ansatz for the particular part of the solution would be

$$y_p = A$$

Which, when substituted into the differential equation, gives $A = d/c$. The full solution is therefore

$$y(t) = c_1 y_1 + c_2 y_2 + \frac{d}{c}$$

By problem 38 of Section 3.5, $y(t) \to d/c$ as $t \to \infty$.

If $c = 0$, the solutions to the characteristic equation are

$$r = 0, -b/a$$

so the homogeneous part of the solution is $C_1 + C_2 \mathrm{e}^{-(b/a)t}$. In this case, the ansatz for the particular part of the solution needs to be multiplied by t,

$$y_p = At$$

Substitution into the DE gives us that $y_p = \dfrac{d}{b} t$

Finally, if $b = 0$ as well, the differential equation just becomes

$$ay'' = d \quad \Rightarrow \quad y'' = \frac{d}{a} \quad \Rightarrow \quad y' = \frac{d}{a} t + C_1 \quad \Rightarrow \quad y = \frac{d}{2a} t^2 + C_1 t + C_2$$

We see that the homogeneous part of the solution is $C_1 t + C_2$ and the particular part of the solution becomes

$$y_p(t) = \frac{d}{2a} t^2$$