Solutions to Selected Problems, 3.8

1. In problems 1-4, the only tricky part is to get δ. Recall that we want a four-quadrant inverse tangent.

   (a) Problem 1: The point $(3,4)$ is in the first quadrant, so

   $$R=\sqrt{9+16}=5 \qquad \delta=\tan^{-1}(4/3)\approx 0.9273 \text{ rad}$$

   (b) Problem 2: The point $(-1,\sqrt{3})$ is in the second quadrant, so:

   $$R=\sqrt{1+3}=2 \qquad \delta=\tan^{-1}(-\sqrt{3})+\pi=-\pi/3+\pi=2\pi/3$$

   (c) Problem 3: The point $(4,-2)$ is in the fourth quadrant, so:

   $$R=\sqrt{16+4}=2\sqrt{5} \qquad \delta=\tan^{-1}(-1/2)\approx -0.4636 \text{ rad}$$

2. Problem 6: *Note* that there may be some confusion over which units to use. On quizzes/exams, we will always use the standard units of meters, kilograms and seconds, or feet, pounds and seconds.

   The constants in this problem (the spring constant is found using the relation: $mg - kL = 0$, or $k = mg/L$):

   - Mass: 100 grams or 0.1 kg
   - Gravity: 9.8 meters/second^2
   - Length: 5 cm or 0.05 meters
   - Spring constant: $k = (9.8)(0.1)/0.05 = 19.6$
   - The initial velocity: 10 cm or 0.1 meters

   So, using meters, kg, seconds:

   $$0.1u'' + 19.6u = 0 \text{ or } u'' + 196u = 0 \qquad r = 14$$

   $$u = C_1\cos(14t) + C_2\sin(14t) \qquad u(0)=0 \quad u'(0)=0.1$$

   so that

   $$u=\frac{1}{140}\sin(14t) \text{ in meters, or } \quad \frac{5}{7}\sin(14t) \text{ cm}$$

   In either case, the time (in seconds) to equilibrium:

   $$\sin(14t)=0 \quad \Rightarrow \quad 14t = 0,\pi,2\pi,\ldots$$

   We want the first time we return to equilibrium, so $14t=\pi$, or $t=\pi/14$.

3. Problem 12: From the discussion on P. 201-202, we have:

$$LQ'' + RQ' + \frac{1}{C}Q = 0$$

with $L = 2 \times 10^{-1}$, $R = 3 \times 10^2$, and $1/C = 1/10^{-5} = 10^5$. Therefore,

$$2 \times 10^{-1}Q'' + 3 \times 10^2 Q' + 10^5 Q = 0 \text{ or } 2Q'' + 3 \times 10^3 Q' + 10^6 Q = 0$$

From the characteristic equation:

$$r = \frac{-3 \times 10^3 \pm \sqrt{9 \times 10^6 - 8 \times 10^6}}{4} = 10^3 \left(\frac{-3 \pm 1}{4}\right) = -500, -1000$$

Therefore,

$$Q = C_1 \mathrm{e}^{-500t} + C_2 \mathrm{e}^{-1000t}$$

The initial conditions translate to: $Q(0) = 10^{-6}$ and $Q'(0) = 0$. give $C_1 = 2 \times 10^{-6}$ and $C_2 = -10^{-6}$, or

$$Q = 10^{-6}\left(2\mathrm{e}^{-500t} - \mathrm{e}^{-1000t}\right)$$

4. Problem 13: The quasi-period of the damped motion is 50% greater than the period of the undamped motion, so we need to find the period of the undamped motion:

$$u'' + u = 0 \quad \Rightarrow \quad u = A\cos(t) + B\sin(t)$$

The period is $2\pi/1$, or 2π. The quasi-period of the damped motion is then 3π. If we assume a form of $R\cos(\omega t - \delta)$, then:

$$\frac{2\pi}{\omega} = 3\pi \quad \Rightarrow \quad \omega = \frac{2}{3}$$

The roots to the characteristic equation of the damped motion are:

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2}$$

If we want to get quasi-periodic motion, then $\gamma^2 - 4 < 0$, so we'll write $\sqrt{\gamma^2 - 4}$ as $i\sqrt{4 - \gamma^2}$, and:

$$r = -\frac{\gamma}{2} \pm \frac{\sqrt{4 - \gamma^2}}{2} = \lambda \pm \omega i$$

We want $\omega = \frac{2}{3}$, so:

$$\frac{\sqrt{4 - \gamma^2}}{2} = \frac{2}{3} \quad \Rightarrow \quad \gamma^2 = 4 - \frac{16}{9} = \frac{20}{9}$$

so $\gamma = \frac{2\sqrt{5}}{3}$.

5. Problem 14: The period of motion from an undamped system is from the solution to:

$$mu'' + ku = 0 \quad \Rightarrow \quad u = R\cos\left(\sqrt{\frac{k}{m}}t - \delta\right)$$

The period is:

$$\frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}$$

Recall that $mg - kL = 0$, or $k = mg/L$. Replacing k in the above equation gives us the result,

$$2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{mL}{mg}} = 2\pi\sqrt{\frac{L}{g}}$$

6. Problem 15: By the linearity of the differential equation, if v, w each solves $mu'' + \gamma u' + ku = 0$, then so does $v + w$ (in fact, any linear combination $c_1 v + c_2 w$ solves it).

Therefore, we just need to check that $u = v + w$ satisfies the initial conditions:

$$u(t_0) = v(t_0) + w(t_0) = u_0 + 0 = u_0$$
$$u'(t_0) = v'(t_0) + w'(t_0) = 0 + u_0' = u_0'$$

7. Problem 24: Consider

$$\frac{3}{2}u'' + ku = 0 \qquad u(0) = 2 \quad u'(0) = v$$

We want to determine k, v so that the amplitude of our solution is 3 and the period is π. First get the solution by solving the characteristic equation:

$$\frac{3}{2}r^2 + k = 0 \quad \Rightarrow \quad r = \pm\sqrt{\frac{2k}{3}}$$

Therefore, the period of our solution will be:

$$\text{Period} = \frac{2\pi}{\sqrt{\frac{2k}{3}}} = \pi$$

Solve this for k to get that $k = 6$, which means that $r = \pm 2i$.

Our solution is now:

$$u = C_1\cos(2t) + C_2\sin(2t)$$

Using the initial conditions, $C_1 = 2$ and $C_2 = \frac{v}{2}$

The amplitude can be determined now in terms of v:

$$R^2 = C_1^2 + C_2^2 \quad \Rightarrow \quad 9 = 4 + \frac{v^2}{4}$$

we get that $v^2 = 20$, or $v = \pm 2\sqrt{5}$.

8. Problem 26: Probably best to leave in general constants until the very end.

   The solutions to the characteristic equation are:

$$r = -\frac{\gamma}{2m} \pm \frac{\sqrt{4km - \gamma^2}}{2m} i \doteq \lambda \pm \mu i$$

   so that the general solution is:

$$u = \mathrm{e}^{\lambda t} \left(c_1 \cos(\mu t) + c_2 \sin(\mu t) \right)$$

   Solve using the initial conditions $u(0) = u_0, u'(0) = v_0$, we get:

$$c_1 = u_0 \qquad c_2 = \frac{v_0 - \lambda u_0}{\mu}$$

   so that the (squared) amplitude is:

$$R^2 = c_1^2 + c_2^2 = u_0^2 + \left(\frac{v_0 - \lambda u_0}{\mu} \right)^2$$

   We substitute the values in for λ and μ noticing that

$$\frac{1}{\mu^2} = \frac{4m^2}{4km - \gamma^2}$$

   Now:

$$R^2 = u_0^2 + \frac{(v_0 + \frac{\gamma}{2m} u_0)^2 \cdot 4m^2}{4km - \gamma^2} = u_0^2 + \frac{(2mv_0 + \gamma u_0)^2}{4km - \gamma^2}$$

   This answer is equivalent to the text's answer, but I think it's easier to read.

   In any event, it is clear that, as $\gamma \to 4km$, the amplitude increases (the period increases as well).