Selected Solutions, Section 5.1

1. Problem 8: Use the Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)!|x|^{n+1}}{(n+1)^{n+1}} \frac{n^n}{|x|^n n!} = |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

In class, we talked about the technique where we exponentiate to use L'Hospital's rule:

$$\left(\frac{n}{n+1}\right)^n = \mathrm{e}^{n\ln\left(\frac{n}{n+1}\right)}$$

so now we take the limit of the exponent:

$$\lim_{n\to\infty} n\ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}$$

which is of the form 0/0. Continue with L'Hospital:

$$\lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} \stackrel{L}{=} \lim_{n\to\infty} \frac{\frac{n+1}{n}\cdot\frac{n+1-n}{(n+1)^2}}{-\frac{1}{n^2}} == \lim_{n\to\infty} \frac{1}{n(n+1)}\cdot\frac{-n^2}{1} = \lim_{n\to\infty} \frac{-n}{n+1} = -1$$

Therefore,

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \mathrm{e}^{n\ln\left(\frac{n}{n+1}\right)} = \mathrm{e}^{-1}$$

And the ratio test:

$$\frac{|x|}{e} < 1 \quad \Rightarrow \quad |x| < \mathrm{e}$$

2. Problem 12: Actually, this is kind of a "trick question", although the usual procedure still works:

$$f(x) = x^2 \quad \Rightarrow \quad f(-1) = 1$$
$$f'(x) = 2x \quad \Rightarrow \quad f'(-1) = -2$$
$$f''(x) = 2 \quad \Rightarrow \quad f''(-1) = 2$$

Therefore,

$$x^2 = 1 - 2(x+1) + \frac{2}{2!}(x+1)^2 = 1 - 2(x+1) + (x+1)^2$$

(Notice that if you expand and simplify this, you get x^2 back.)

This is not an infinite series; no matter what x is, you can always add those three terms together: The radius of convergence is ∞.

3. Problem 14. At issue here is to find a pattern in the derivatives, so we can write the general form for the n^{th} derivative.

$$\begin{array}{lll} n=0 & f(x)=(1+x)^{-1} & f(0)=1 \\ n=1 & f'(x)=-(1+x)^{-2} & f'(0)=-1 \\ n=2 & f''(x)=(-1)(-2)(1+x)^{-3} & f''(0)=2 \\ n=3 & f'''(x)=(-1)(-2)(-3)(1+x)^{-4} & f'''(0)=-3! \end{array}$$

From this we see that:

$$f^{(n)}(0)=(-1)^n n!$$

The Taylor series (actually, the Maclaurin series) is:

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}\frac{(-1)^n n!}{n!}x^n=\sum_{n=0}^{\infty}(-x)^n$$

and this converges if $|x|<1$ (its an alternating geometric series).

For extra practice, we could see this directly using the sum of the geometric series:

$$\sum_{n=0}^{\infty}(-x)^n=\frac{1}{1-(-x)}=\frac{1}{1+x}$$

4. Problem 16. Very similar to 14, we try to find the pattern in the derivatives.

$$\begin{array}{lll} n=0 & f(x)=(1-x)^{-1} & f(2)=-1 \\ n=1 & f'(x)=(1-x)^{-2} & f'(2)=1 \\ n=2 & f''(x)=(1)(2)(1-x)^{-3} & f''(2)=-2 \\ n=3 & f'''(x)=(1)(2)(3)(1-x)^{-4} & f'''(2)=3! \end{array}$$

The pattern is: $f^{(n)}(2)=(-1)^{n+1}n!$. Therefore, the Taylor series is:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}n!}{n!}(x-2)^n=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$$

with a radius of convergence $\rho=1$.

5. Problem 18: Given that

$$y=\sum_{n=0}^{\infty}a_n x^n$$

Compute y' and y'' by writing out the first four terms of each to get the general term. Show that, if $y''=y$, then the coefficients a_0 and a_1 are arbitrary, and show the given recursion relation.

$$y=a_0+a_1x+a_2x^2+a_3xx^3+\ldots=\sum_{n=0}^{\infty}a_n x^n$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y'' = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \ldots = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

If $y'' = y$, then the coefficients must match up, power by power:

$$a_0 = 2a_2 \qquad a_1 = 6a_3 \qquad a_2 = 12a_4 \qquad \ldots \qquad a_n = (n+2)(n+1)a_{n+2}$$

Problems 19-23 are some symbolic manipulation problems.

6. Problem 19: Rewrite the left side equation so that the powers of x match up.

7. Problem 20: Much the same. In this problem, we see that the first sum starts with a constant term, the second sum starts with x^1, and so does the sum on the left. Therefore, we would rewrite each sum to start with x^1 power:

$$\sum_{k=1}^{\infty} a_{k+1}x^k = a_1 + \sum_{n=1}^{\infty} a_{n+1}x^n$$

$$\sum_{k=0}^{\infty} a_kx^{k+1} = \sum_{n=1}^{\infty} a_{n-1}x^n$$

Now each sum begins with the same power of x,

$$\sum_{k=1}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_kx^{k+1} = a_1 + \sum_{n=1}^{\infty} a_{n+1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = a_1 + \sum_{n-0}^{\infty}\left(a_{n+1} + a_{n-1}\right)x^n$$

8. Problem 21: You may use a different symbol for the summation index if you like (it is a dummy variable):

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$$

We would like this to be indexed using x^k, $k = 0, 1, 2, \ldots$. This means that $k = n-2$ or $n = k+2$. Making the substitutions in each term,

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k$$

9. Problem 22: In this case, the powers begin with x^2, so we let $k = n+2$ or $n = k-2$, with $k = 2, 3, 4, \ldots$:

$$\sum_{n=0}^{\infty} a_nx^{n+2} = \sum_{k=2}^{\infty} a_{k-2}x^k$$

10. Problem 23: Take care of the product with x first,

$$x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k = \sum_{n=1}^{\infty} na_n x^n + \sum_{k=0}^{\infty} a_k x^k$$

The first sum could begin with zero- It would make the first term of the sum zero. Therefore,

$$\sum_{n=0}^{\infty} na_n x^n + \sum_{k=0}^{\infty} a_k x^k = \sum_{n=1}^{\infty} (n+1)a_n x^n$$