Selected Solutions, Section 5.3

Since we are running short of time for Chapter 5, we'll delete exercises 12, 16-18 from the homework.

1. Problem 1: We determine the derivatives by simply differentiating and evaluating at the given point. We will go ahead and use $y(x)$ in place of $\phi(x)$. Technically speaking, these are not the same thing (ϕ is the series approximation to the true solution y):

$$y(0) = 1 \qquad y'(0) = 0$$
$$y'' = -xy' - y \quad \Rightarrow y''(0) = -1$$
$$y''' = -y' - xy'' - y' = -2y' - xy'' \Rightarrow y'''(0) = 0$$
$$y^{iv} = -2y'' - y'' - xy''' = -3y'' - xy''' \Rightarrow y^{iv}(0) = 3$$

2. Problem 3:

$$y(1) = 2 \qquad y'(1) = 0$$
$$x^2y'' + (1+x)y' + 3\ln(x)y = 0 \Rightarrow y'' + 2(0) + 3(0)(2) = 0 \Rightarrow y'' = 0$$

Probably best to just differentiate in place, simplify, then evaluate at $x = 1$:

$$2xy'' + x^2y''' + y' + (1+x)y'' + \frac{3}{x}y + 3\ln(x)y' = 0 \Rightarrow$$
$$x^2y''' + (1+3x)y'' + (1+3\ln(x))y' + \frac{3y}{x} = 0 \Rightarrow$$
$$y''' + 4(0) + 0 + 6 = 0 \Rightarrow y'''(1) = -6$$

Similarly, for the fourth derivative:

$$2xy''' + x^2y^{iv} + 3y'' + (1+3x)y''' + \frac{3}{x}y' + (1+3\ln(x))y'' - \frac{3}{x^2}y + \frac{3}{x}y' = 0$$

so that $y^{iv}(1) = 42$.

3. Problem 5: In problem 5, $p(x) = 4$ and $q(x) = 6x$. These functions have a radius of convergence of ∞, so we expect that, no matter the base point, the solution will also have a radius of convergence of ∞.

4. Problem 6: In this case,

$$p(x) = \frac{x}{x^2 - 2x - 3} = \frac{x}{(x-3)(x+1)} \qquad q(x) = \frac{4}{(x-3)(x+1)}$$

Both p and q fail to exist at $x = 3$ and $x = -1$. Therefore, we expect that these points will not be included in the interval of convergence for $y(x)$. It is perhaps easiest to construct a number line:

$$x_0 = -4 \qquad -1 \qquad x_0 = 0 \qquad 3 \qquad x_0 = 4$$

- For $x_0 = -4$, the closest "bad point" is -1, which is 3 units away. Therefore, in this case we expect the radius of convergence to be 3.
- For $x_0 = 0$, the closest bad point is -1, which is 1 unit away. The radius of convergence would be 1.
- For $x_0 = 4$, the closest bad point is 3, which is again 1 unit away, so the radius of convergence is 1.

Problems 11 and 12 are very similar to Problems 1 and 3. The series is found by differentiating and substituting in the particular values of x. Notice that, to get the linearly independent functions, first set $y(0) = 1$, $y'(0) = 0$, then $y(0) = 0$ and $y'(0) = 1$ (this is equivalent to setting $a_0 = 1, a_1 = 0$, then taking $a_0 = 0, a_1 = 1$ from the previous section).

5. Problem 11: For the radius of convergence, we expect ∞, since $p(x) = 0$ and $q(x) = \sin(x)$ are analytic for all x.

   We differentiate a few times, then evaluate at $x = 0$:

$$y'' = -\sin(x)y$$

$$\begin{aligned}
y''' &= -\cos(x)y - \sin(x)y' \\
y^{iv} &= \sin(x)y - 2\cos(x)y' - \sin(x)y'' \\
y^{v} &= \cos(x)y + 3\sin(x)y' - 3\cos(x)y'' - \sin(x)y''' \\
y^{vi} &= -\sin(x)y + 4\cos(x)y' + 6\sin(x)y'' - 4\cos(x)y''' - \sin(x)y^{iv} \\
y^{vii} &= -\cos(x)y - 5\cos(x)y' + 10\sin(x)y'' + 10\cos(x)y''' - 5\cos(x)y^{iv} - \sin(x)y^{v}
\end{aligned}$$

   Cool! Do you see Pascal's Triangle?

   We'll simplify (a lot) by taking $x = 0$:

	$y(0) = 1, y'(0) = 0$	$y(0) = 0, y'(0) = 1$
$y'' = 0$	0	0
$y''' = -y$	-1	0
$y^{(4)} = -2y'$	0	-2
$y^{(5)} = y - 3y'' = y$	1	0
$y^{(6)} = 4y' - 4y'''$	4	4
$y^{(7)} = -y + 10y'' - 5y^{(4)} = -y - 5y^{(4)}$	-1	10

   so:

$$y_1(x) = 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{4}{6!}x^6 - \frac{1}{7!}x^7 + \dots$$

$$y_2(x) = x - \frac{2}{4!}x^4 + \frac{4}{6!}x^6 + \frac{10}{7!}x^7 + \dots$$

6. Problem 12: In this case, we also expect the radius of convergence to be infinite, since $q(x) = x\mathrm{e}^{-x}$ is analytic for all x.

   As in problem 11, differentiate and evaluate the derivatives at the origin:

$$\mathrm{e}^x y'' + xy = 0$$

$$\begin{aligned} \mathrm{e}^x y''' + \mathrm{e}^x y'' + xy' + y = 0 \\ \mathrm{e}^x y^{(4)} + 2\mathrm{e}^x y''' + \mathrm{e}^x y'' + xy'' + 2y' = 0 \\ \mathrm{e}^x y^{(5)} + 3\mathrm{e}^x y^{(4)} + 3\mathrm{e}^x y''' + \mathrm{e}^x y'' + xy''' + 3y'' = 0 \end{aligned}$$

   Evaluating at $x = 0$,

	$y(0) = 1, y'(0) = 0$	$y(0) = 0, y'(0) = 1$
$y'' = 0$	0	0
$y''' = -y'' - y$	-1	0
$y^{(4)} = -2y''' - y'' - 2y'$	2	-2
$y^{(5)} = -3y^{(4)} - 3y''' - 4y''$	-3	6

   so that:

$$y_1(x) = 1 - \frac{1}{3!}x^3 + \frac{2}{4!}x^4 - \frac{3}{5!}x^5 + \ldots$$

$$y_2(x) = x - \frac{2}{4!}x^4 + \frac{6}{5!}x^5 + \ldots$$

7. Problem 15: Using a different argument from the text, if x, x^2 are solutions to the differential equation, we can substitute them in and see what we get.

   For $y = x$, $P(x)y'' + Q(x)y' + R(x)y = 0$ becomes:

$$Q(x) = -xR(x)$$

   For $y = x^2$, the differential equation becomes:

$$2P(x) + 2xQ(x) + x^2R(x) = 0 \quad \Rightarrow \quad 2P + 2x(-xR) + x^2R = 0 \quad \Rightarrow \quad P = -\frac{x^2}{2}R$$

   Substituting these in,

$$p(x) = \frac{Q(x)}{P(x)} = \frac{2xR(x)}{x^2R(x)} = \frac{2}{x}$$

   and

$$q(x) = \frac{R(x)}{P(x)} = \frac{-2R(x)}{x^2R(x)} = -\frac{2}{x^2}$$

   so that the point $x = 0$ is a singular point (not an ordinary point).