Selected Solutions, Section 6.4

1. Problem 2:
$$y'' + 2y' + 2y = u_\pi(t) - u_{2\pi}(t), \quad y(0) = 0, \quad y'(0) = 1$$
Take the Laplace transforms and solve for $Y(s)$:
$$(s^2Y - 0 - 1) + 2(sY - 0) + 2Y = \left(e^{-\pi s} - e^{-2\pi s}\right)\frac{1}{s}$$
$$(s^2 + 2s + 2)Y = \left(e^{-\pi s} - e^{-2\pi s}\right)\frac{1}{s} + 1$$
$$Y = \left(e^{-\pi s} - e^{-2\pi s}\right)\frac{1}{s(s^2 + 2s + 2)} + \frac{1}{s^2 + 2s + 2}$$
We'll do the last term first:
$$\frac{1}{s^2 + 2s + 2} = \frac{1}{s^2 + 2s + 1 + 1} = \frac{1}{(s + 1)^2 + 1}$$
so the inverse Laplace transform is (table entry 19): $e^{-t}\sin(t)$.
Next, notice that the first term is of the form:
$$\left(e^{-\pi s} - e^{-2\pi s}\right)\frac{1}{s(s^2 + 2s + 2)} = \left(e^{-\pi s} - e^{-2\pi s}\right)H(s)$$
So if we find $h(t)$, the inverse Laplace transform of this part will be:
$$u_\pi(t)h(t - \pi) - u_{2\pi}(t)h(t - 2\pi)$$
Therefore, we only need to focus on inverting:
$$H(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$
Solving, we get $A = 1/2$, $B = -1/2$, $C = -1$, or:
$$\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s + 2}{s^2 + 2s + 2} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\left[\frac{s + 1}{(s + 1)^2 + 1} + \frac{1}{(s + 1)^2 + 1}\right]$$
Now $h(t) = \frac{1}{2} - \frac{1}{2}\left(e^{-t}\cos(t) + e^{-t}\sin(t)\right)$. Putting it all together,
$$y(t) = e^{-t}\sin(t) + u_\pi(t)h(t - \pi) - u_{2\pi}(t)h(t - 2\pi)$$
See the Maple worksheet for verification and the plots.

2. Problem 6:
$$y'' + 3y' + 2y = u_2(t), \qquad y(0) = 0, \quad y'(0) = 1$$
As is usual, take the Laplace transforms and solve for $Y(s)$:
$$(s^2Y - 0 - 1) + 3(sY - 0) + 2Y = e^{-2s}\frac{1}{s}$$
$$Y = e^{-2s}\frac{1}{s(s^2 + 3s + 2)} + \frac{1}{s^2 + 3s + 2}$$
Break it up- Let's do the second term first:
$$\frac{1}{s^2 + 3s + 2} = \frac{1}{(s + 2)(s + 1)} = \frac{A}{s + 2} + \frac{B}{s + 1} = -\frac{1}{s + 2} + \frac{1}{s + 1}$$

so the inverse Laplace transform of this part is: $-e^{-2t} + e^{-t}$.

Alternative Solution to this part. We could have completed the square in the denominator:

$$\frac{1}{s^2+3s+2} = \frac{1}{s^2+3s+\frac{9}{4}+2-\frac{9}{4}} = 2\frac{\frac{1}{2}}{\left(s+\frac{3}{2}\right)^2 - \frac{1}{4}}$$

Combine table entries 14 and 7 to get the inverse Laplace transform as:

$$2e^{-\frac{3}{2}t}\sinh\left(\frac{1}{2}t\right)$$

This is the solution that Maple gives you. Notice that it is the same as our solution:

$$2e^{-\frac{3}{2}t}\sinh\left(\frac{1}{2}t\right) = 2e^{-\frac{3}{2}t}\cdot\frac{1}{2}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right) = e^{-t} - e^{-2t}$$

It is probably easier to factor and use partial fractions...

3. Problem 15:

We want a function g that:

- Ramps up from the point $(t_0, 0)$ to (t_0+k, h)
- Then ramps down from (t_0+k, h) to $(t_0+2k, 0)$.

Notice that this is simply two line segments. We have pairs of points for each, so let's write the equation of each line segment:

- Slope: $\frac{h}{k}$, so: $y - 0 = \frac{h}{k}(t - t_0)$.
- Slope: $\frac{-h}{k}$, so: $y - 0 = -\frac{h}{k}(t - (t_0 + 2k))$

Finally, use the "On-Off" switch for each line segment. The first line segment comes on at time t_0, off at $t_0 + k$):

$$(u_{t_0}(t) - u_{t_0+k}(t))\left(\frac{h}{k}(t-t_0)\right)$$

We'll add in the second switch,

$$(u_{t_0+k}(t) - u_{t_0+2k}(t))\left(-\frac{h}{k}(t-t_0-2k)\right)$$

Add everything together. To get the answer in the text, factor the slope out and expand (for us, you can leave your answer in this form).

$$(u_{t_0}(t) - u_{t_0+k}(t))\left(\frac{h}{k}(t-t_0)\right) + (u_{t_0+k}(t) - u_{t_0+2k}(t))\left(-\frac{h}{k}(t-t_0-2k)\right)$$

Alternatively, you could also have written the second line segment using the first ordered pair.

4. Problem 19: Straightforward to write down, a little tricky to analyze:

$$y'' + y = u_0(t) + 2\sum_{k=1}^{n} 2(-1)^k u_{k\pi}(t)$$

Taking the Laplace transform, solve for Y and inverting:

$$(s^2+1)Y = \frac{1}{s} + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}\frac{1}{s} \quad\Rightarrow\quad Y = \frac{1}{s(s^2+1)} + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}\frac{1}{s(s^2+1)}$$

where

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

so that

$$y(t) = 1 - \cos(t) + 2\sum_{k=1}^{n} (-1)^k u_{k\pi t} \left(1 - \cos(t - k\pi)\right)$$

This is a bit difficult to analyze in this form. However, if we consider the graph of the cosine, we see that:

If k is odd	$\cos(t - k\pi) = -\cos(t)$
If k is even	$\cos(t - k\pi) = \cos(t)$

Here we write out the function in the form of a table:

kth term		Term is active:
$k=1$	$-2(1+\cos(t))$	$t \ge \pi$
$k=2$	$2(1-\cos(t))$	$t \ge 2\pi$
$k=3$	$-2(1+\cos(t))$	$t \ge 3\pi$
$k=4$	$2(1-\cos(t))$	$t \ge 4\pi$
$\vdots$	$\vdots$	

Therefore, writing $y(t)$ in piecewise form (for clarity, I'm writing it as a table):

Interval	$y(t)$
$t < \pi$	$-\cos(t) + 1$
$\pi \le t < 2\pi$	$(1 - \cos(t) - 2(1 + \cos(t)) = -3\cos(t) - 1$
$2\pi \le t < 3\pi$	$3(1 - \cos(t)) - 2(1 + \cos(t)) = -5\cos(t) + 1$
$3\pi \le t < 4\pi$	$3(1 - \cos(t)) - 4(1 + \cos(t)) = -7\cos(t) - 1$
$4\pi \le t < 5\pi$	$5(1 - \cos(t)) - 4(1 + \cos(t)) = -9\cos(t) + 1$

5. Problem 21 is similar to Problem 19. Here the solution is

$$y(t) = (1 - \cos(t)) + \sum_{k=1}^{n} (-1)^k u_{k\pi}(t)(1 - \cos(t - k\pi))$$

Writing this solution down piecewise (see the pattern?):

Interval	$y(t)$
$t < \pi$	$1 - \cos(t)$
$\pi \le t < 2\pi$	$(1 - \cos(t)) - (1 + \cos(t)) = -2\cos(t)$
$2\pi \le t < 3\pi$	$2(1 - \cos(t)) - (1 + \cos(t)) = -3\cos(t) + 1$
$3\pi \le t < 4\pi$	$2(1 - \cos(t)) - 2(1 + \cos(t)) = -4\cos(t)$
$4\pi \le t < 5\pi$	$3(1 - \cos(t)) - 2(1 + \cos(t)) = -5\cos(t) + 1$
$\vdots$	$\vdots$