Selected Solutions, Section 6.5

For the graphs of 1-8, see the Maple Worksheet on our class website.

1. Problem 1: $y'' + 2y' + 2y = \delta(t-\pi), y(0) = 1, y'(0) = 0$

$$s^2Y - s + 2(sY - 1) + 2Y = \mathrm{e}^{-\pi s}$$

$$(s^2 + 2s + 2)Y = \mathrm{e}^{-\pi s} + s + 2 \quad \Rightarrow \quad Y = \mathrm{e}^{-\pi s}\frac{1}{s^2+2s+2} + \frac{s+2}{s^2+2s+2}$$

We do each term separately:

$$\text{Let } H(s) = \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1} \qquad \text{then} \qquad h(t) = \mathrm{e}^{-t}\sin(t)$$

The inverse Laplace transform of the first term is then $u_\pi(t)h(t-\pi)$.
Similarly,

$$\frac{s+2}{s^2+2s+2} = \frac{s+1+1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

so the inverse Laplace here is $\mathrm{e}^{-t}(\cos(t) + \sin(t))$.
The overall solution: $y(t) = \mathrm{e}^{-t}(\cos(t) + \sin(t)) + u_\pi(t)\left(\mathrm{e}^{-(t-\pi)}\sin(t-\pi)\right)$

2. Problem 2: $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$, zero ICs.

$$(s^2+4)Y = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right) \quad \Rightarrow \quad Y = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)\frac{1}{s^2+4}$$

Notice that, if $H(s) = \frac{1}{s^2+4}$, and we find $h(t)$, then the solution is $u_\pi(t)h(t-\pi) - u_{2\pi}(t)h(t-2\pi)$.
Solving for $h(t)$:

$$H(s) = \frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+4} \qquad h(t) = \frac{1}{2}\sin(2t)$$

Just for practice, you might write the overall solution,

$$y(t) = \frac{1}{2}\left(u_\pi(t)\sin(2(t-\pi)) - u_{2\pi}(t)\sin(2(t-2\pi))\right)$$

as a piecewise defined function. Notice that $\sin(2t)$ has a period of π.

3. Problem 3: $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t)$ with ICs $y(0) = 0, y'(0) = \frac{1}{2}$.

$$s^2Y - s0 - \frac{1}{2} + 3(sY - 0) + 2Y = \mathrm{e}^{-5s} + \frac{\mathrm{e}^{-10s}}{s} \quad \Rightarrow \quad Y = \frac{\mathrm{e}^{-5s}}{s^2+3s+2} + \frac{\mathrm{e}^{-10s}}{s(s^2+3s+2)} + \frac{1/2}{s^2+3s+2}$$

The two functions we need to invert:

$$H_1(s) = \frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)} = \frac{1}{s+2} - \frac{1}{s+1} \quad \Rightarrow \quad h_1(t) = \mathrm{e}^{-2t} - \mathrm{e}^{-t}$$

and

$$H_2(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+2)(s+1)} = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s+2} - \frac{1}{s+1} \quad \Rightarrow \quad h_2(t) = \frac{1}{2} + \frac{1}{2}\mathrm{e}^{-2t} - \mathrm{e}^{-t}$$

With this notation, its easy to write the solution:

$$y(t) = \frac{1}{2}h_1(t) + u_5(t)h_1(t-5) + u_{10}(t)h_2(t-10)$$

4. Problem 4: $y'' - y = -20\delta(t-3)$, $y(0) = 1, y'(0) = 0$.

$$s^2Y - s - Y = -20\mathrm{e}^{-3s} \quad \Rightarrow \quad Y = -20\mathrm{e}^{-3s}\frac{1}{s^2-1} + \frac{s}{s^2-1}$$

Note that we could factor $s^2 - 1$ as $(s+1)(s-1)$ and perform partial fractions, but in this case we can use Table Entries 7, 8 directly:

$$y(t) = \cosh(t) - 20u_3(t)\sinh(t-3)$$

5. Problem 5: $y'' + 2y' + 3y = \sin(t) + \delta(t - 3\pi)$, $y(0) = 0, y'(0) = 0$.

$$(s^2 + 2s + 3)Y = \frac{1}{s^2+1} + e^{-3\pi s} \quad \Rightarrow \quad Y = \frac{1}{(s^2+1)(s^2+2s+3)} + e^{-3s}\frac{1}{s^2+2s+3}$$

We'll take the partial fractions on first:

$$H_1(s) = \frac{1}{(s^2+1)(s^2+2s+3)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+3} = -\frac{1}{4}\frac{s-1}{s^2+1} + \frac{1}{2}\frac{s+1}{s^2+2s+3} =$$

$$-\frac{1}{4}\left(\frac{s}{s^2+1} - \frac{1}{s^2+1}\right) + \frac{1}{2}\frac{s+1}{(s+1)^2+2} \quad \Rightarrow \quad h_1(t) = -\frac{1}{4}\left(\cos(t) - \sin(t)\right) + \frac{1}{2}e^{-t}\cos(\sqrt{2}t)$$

For the second term, let

$$H_2(t) = \frac{1}{s^2+2s+3} = \frac{1}{(s+1)^2+2} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2} \quad \Rightarrow \quad h_2(t) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$$

Overall, the solution is then:

$$y(t) = h_1(t) + u_{3\pi}(t)h_2(t - 3\pi)$$

As a double-check, you might notice that before time 3π, $-\frac{1}{4}\left(\cos(t) - \sin(t)\right)$ is the particular part of the solution, and $\frac{1}{2}e^{-t}\cos(\sqrt{2}t)$ is the homogeneous part of the solution.

6. Problems 6, 8 are very similar to 1-5.

7. For problems 17-21, notice that striking the system will "activate" the homogeneous solution, which is otherwise 0. In this case, the homogeneous solution is $C_1\sin(t) + C_2\cos(t)$, which is periodic with period 2π. We show what you could do for a quick hand-sketched analysis on the website.