Selected Solutions, Section 6.6

1. Problem 1:

   (a) We also did this one in class. It is shown via a change of variables:

   $$f * g = \int_0^t f(t-x)g(x)\,dx$$

   Let $u = t - x$, so that $du = -dx$. If $x = 0, u = t$ and if $x = t$, $u = 0$:

   $$f * g = \int_0^t f(t-x)g(x)\,dx = -\int_t^0 f(u)g(t-u)\,du = \int_0^t g(t-u)f(u)\,du = g * f$$

   (b)

   $$f * (g_1 + g_2) = \int_0^t f(t-x)\left(g_1(x) + g_2(x)\right)\,dx =$$

   $$\int_0^t f(t-x)g_1(x) + f(t-x)g_2(x)\,dx =$$

   $$\int_0^t f(t-x)g_1(x)\,dx + \int_0^t f(t-x)g_2(x)\,dx = f * g_1 + f * g_2$$

   (c) Part (c): It was hard to typeset. See the solution as a separate webpage (actually, it's an image file).

2. Problem 2. You can choose almost any function, even $1 * 1$:

   $$1 * 1 = \int_0^t 1\,dx = \quad x\Big|_0^t = t$$

3. Problem 3. The following trig identity is used:

   $$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

   Then:

   $$\sin(t) * \sin(t) = \int_0^t \sin(t-x)\sin(x)\,dx = \frac{1}{2}\int_0^t \cos(t-2x) - \cos(t)\,dx$$

   Use $u = t - 2x$, $du = -2\,dx$ for the first term:

   $$\int \cos(t-2x)\,dx = -\frac{1}{2}\int \cos(u)\,du = -\frac{1}{2}\sin(t-2x)$$

   The full antiderivative becomes:

   $$\frac{1}{2}\left(-\frac{1}{2}\sin(t-2x) - x\cos(t)\Big|_0^t = \frac{1}{2}\left[\left(-\frac{1}{2}\sin(-t) - t\cos(t)\right) - \left(-\frac{1}{2}\sin(t)\right)\right]\right.$$

   from which we get the textbook's answer: $(1/2)(\sin(t) - t\cos(t))$

4. Problem 4. We want to write:

$$\int_0^t (t-\tau)^2 \cos(2\tau)\, d\tau \quad \text{as} \quad \int_0^t f(t-\tau)g(\tau)\, d\tau$$

In this case, we see that $f(t) = t^2$, $g(t) = \cos(2t)$, with corresponding Laplace transforms $F(s) = 2/s^3$ and $G(s) = s/(s^2+4)$. By the convolution theorem:

$$\mathcal{L}(f * g) = F(s)G(s) = \frac{2s}{s^3(s^2+4)} = \frac{2}{s^2(s^2+4)}$$

5. Problem 6: Same type as 4:

$$\int_0^t (t-\tau)\mathrm{e}^{\tau}\, d\tau = f * g$$

where $f(t) = t$, $g(t) = \mathrm{e}^t$, and corresponding Laplace transforms: $F(s) = 1/s^2$ and $G(s) = 1/(s-1)$. Therefore,

$$\mathcal{L}(f * g) = F(s)G(s) = \frac{1}{s^2(s-1)}$$

6. Problem 8: The idea here is to write the given expression as $F(s)G(s)$, so that the inverse transform is $f * g$:

$$\frac{1}{s^4(s^2+1)} = F(s)G(s)$$

where

$$F(s) = \frac{1}{s^4} \qquad G(s) = \frac{1}{s^2+1}$$

Notice that to invert $F(s)$, we need to multiply and divide by $3! = 6$. Now,

$$\mathcal{L}^{-1}(F(s)G(s)) = \frac{1}{6}t^3 * \sin(t) = \frac{1}{6}\int_0^t (t-x)^3 \sin(x)\, dx$$

7. Problem 9. Same idea, you might group the s in the numerator with the s^2+4:

$$\frac{s}{(s+1)(s^2+4)} = F(s)G(s) \quad \text{where } F(s) = \frac{1}{s+1} \quad G(s) = \frac{s}{s^2+4}$$

Therefore,

$$\mathcal{L}^{-1}(F(s)G(s)) = f * g = \mathrm{e}^{-t} * \cos(2t) = \int_0^t \mathrm{e}^{-(t-x)} \cos(2x)\, dx$$

8. Problem 13:

$$y'' + \omega^2 y = g(t) \qquad y(0) = 0 \quad y'(0) = 1$$

$$s^2Y - 1 + \omega^2 Y = G(s) \quad \Rightarrow \quad (s^2+\omega^2)Y = G(s)+1 \quad \Rightarrow \quad Y = G(s)\frac{1}{s^2+\omega^2} + \frac{1}{s^2+\omega^2}$$

With:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{\omega}\frac{\omega}{s^2+\omega^2}\right) = \frac{1}{\omega}\sin(\omega t)$$

Therefore, the solution is (the question asked us to write it as an integral):

$$y(t) = \frac{1}{\omega}\left(g(t) * \sin(\omega t) + \sin(\omega t)\right) = \frac{1}{\omega}\left(\int_0^t g(t-x)\sin(\omega x)\,dx + \sin(\omega t)\right)$$

9. Problem 14: Similar to Problem 13,

$$y'' + 2y' + 2y = \sin(\alpha t) \qquad \text{zero ICs}$$

$$(s^2 + 2s + 2)Y = \frac{\alpha}{s^2+\alpha^2} \quad \Rightarrow$$

$$Y = \frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{s^2+2s+2} = \frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{(s+1)^2+1}$$

The inverse transform is the convolution of the inverses taken separately,

$$\mathcal{L}^{-1}\left(\frac{\alpha}{s^2+\alpha^2}\right) = \sin(\alpha t) \qquad \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) = \mathrm{e}^{-t}\sin(t)$$

so that:

$$y(t) = \sin(\alpha t) * \mathrm{e}^{-t}\sin(t) = \int_0^t \sin(\alpha(t-x))\mathrm{e}^{-x}\sin(x)\,dx$$