Homework Solutions (to problems not in the text)

Week of April 23d

1. Find the eigenvalues and eigenvectors for each of the following matrices:

   (a)
$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

   We note that the trace is 1, the determinant is -2 and the discriminant is 9

   The eigenvalues are found by the characteristic equation, coming from the determinant of:

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad \lambda = 2,\ -1$$

   For each λ, find the corresponding eigenvector:

   - For $\lambda = 2$,
$$\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
     so that $v_1 - 2v_2 = 0$ or $v_1 = 2v_2$. Writing it this way leaves v_2 as the free variable,
$$\begin{aligned} v_1 &= 2v_2 \\ v_2 &= v_2 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_1$$
     or (since any multiple of an eigenvector is still an eigenvector):
$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   - For $\lambda = -1$:
$$\begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
     so that $2v_1 - v_2 = 0$ or $v_2 = 2v_1$. Writing it this way leaves v_1 as the free variable,
$$\begin{aligned} v_1 &= v_1 \\ v_2 &= 2v_2 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   (b)
$$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

   The trace is 2, the determinant is 5, and the discriminant is -16. Therefore,
$$\lambda = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

- For $\lambda = 1 - 2i$

$$\begin{bmatrix} 3-1+2i & -2 \\ 4 & -1-1+2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2+2i & -2 \\ 2 & -2+2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  so that $(2+2i)v_1 - 2v_2 = 0$ or $v_2 = (1+i)v_1$. Writing it this way leaves v_1 as the free variable,

$$\begin{aligned} v_1 &= v_1 \\ v_2 &= (1+i)v_1 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix} v_1$$

- You should verify this, but the second eigenvector will be the complex conjugate of the first. For $\lambda = 1 + 2i$, we should get:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

  We can verify this by using the eigenvalue-eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \begin{bmatrix} 3-2+2i \\ 4-1+i \end{bmatrix} = \begin{bmatrix} 1+2i \\ 3+i \end{bmatrix} = (1+2i) \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

The trace is 2, the determinant is 1, and the discriminant is 0. We only get one eigenvalue,

$$\lambda = \frac{2 \pm 0}{2} = 1$$

For the eigenvector,

$$\begin{bmatrix} 3-1 & -4 \\ 1 & -1-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $v_1 - 2v_2 = 0$ or $v_1 = 2v_2$. Writing it this way leaves v_2 as the free variable,

$$\begin{aligned} v_1 &= 2v_2 \\ v_2 &= v_2 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2. For each matrix below, find the eigenvalues in terms of the parameter α. Describe how the eigenvalues change with respect to α.

   (a)

$$\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix} \qquad \mathrm{Tr}(A) = 2\alpha \quad \det(A) = \alpha^2 + 1 \quad \Delta = -4$$

so that

$$\lambda = \frac{2\alpha \pm 2i}{2} = \alpha \pm i$$

The eigenvalues will always be complex (pure imaginary if $\alpha = 0$). This will be more important later on.

(b)

$$\begin{bmatrix} -1 & \alpha \\ -1 & -1 \end{bmatrix} \qquad \text{Tr}(A) = -2 \quad \det(A) = 1 + \alpha \quad \Delta = -\alpha$$

Therefore,

$$\lambda = \frac{-2 \pm \sqrt{-\alpha}}{2}$$

If $\alpha < 0$, the eigenvalues will be distinct real values.
If $\alpha = 0$, we will have a single real eigenvalue.
If $\alpha > 0$, we will have complex conjugate eigenvalues.

3. Show that the given function solves the given differential equation:

(a)

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x} \qquad x = \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t}$$

Differentiating first,

$$\mathbf{x}' = 2e^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} e^{2t}$$

Now we check the right hand side of the equation:

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t} = \begin{bmatrix} 3 \cdot 4 - 2 \cdot 2 \\ 2 \cdot 4 - 2 \cdot 2 \end{bmatrix} e^{2t} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} e^{2t}$$

(b) **ERROR:** There was an exponential function missing from the homework page. The differential equation should read:

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ -e^t \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t$$

To differentiate, we can write the solution in parametric form:

$$\mathbf{x}(t) = \begin{bmatrix} e^t + 2te^t \\ 2te^t \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}'(t) = \begin{bmatrix} 3e^t + 2te^t \\ 2e^t + 2te^t \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}'(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t$$

For the right hand side of the differential equation, first we'll handle the matrix:

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^t + 2te^t \\ 2te^t \end{bmatrix} = \begin{bmatrix} 2e^t + 4te^t - 2te^t \\ 3e^t + 6te^t - 4te^t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t$$

It is not the same yet- It is when we add $< e^t, -e^t >$ to that vector.

4. Given the vector function below, compute $\int \mathbf{x}(t)\,dt$:

$$\mathbf{x}(t) = \begin{bmatrix} \sin(t) \\ t\mathrm{e}^{-3t} \end{bmatrix} \quad \Rightarrow \quad \int \mathbf{x}(t)\,dt = \begin{bmatrix} -\cos(t) \\ -(1/9)(1+3t)\mathrm{e}^{-3t} \end{bmatrix}$$

5. Given that we can integrate componentwise, we can then also compute Laplace transforms. Compute the Laplace transform of:

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x} \qquad \mathbf{x}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

(For notation, you might let $\hat{x}_1$ be the Laplace transform of $x_1(t)$, etc.).

The Laplace transform:

$$s\hat{\mathbf{x}} - \mathbf{x}(0) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \hat{\mathbf{x}}$$

We can solve for $\hat{\mathbf{x}}$ like we did for the eigenvalue-eigenvector equation:

$$s\hat{\mathbf{x}} - \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \hat{\mathbf{x}} = \mathbf{x}(0) \quad \Rightarrow \quad \begin{bmatrix} s-3 & 2 \\ -2 & s+2 \end{bmatrix} \hat{\mathbf{x}} = \mathbf{x}(0)$$

This is further than you needed to go, but for the entertainment value, we can further solve for the Laplace transform using the formula for the inverse of a 2×2 matrix (No, we won't typically use this formula):

$$\hat{\mathbf{x}} = \frac{1}{s^2 - s - 2} \begin{bmatrix} s+2 & -2 \\ 2 & s-3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \frac{1}{(s+1)(s-2)} \begin{bmatrix} 2s+10 \\ -3s+13 \end{bmatrix}$$

We can invert each of these to get:

$$\mathbf{x}(t) = \begin{bmatrix} \frac{14}{3}\mathrm{e}^{2t} - \frac{8}{3}\mathrm{e}^{-t} \\ \frac{7}{3}\mathrm{e}^{2t} - \frac{16}{3}\mathrm{e}^{-t} \end{bmatrix}$$