Quiz 6 Solutions

1. Use the *definition* of the Laplace Transform to compute $\mathcal{L}(f(t))$, where $f(t)$ is given as:

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 1 \\ 2+t & \text{if } t > 1 \end{cases}$$

Be sure to include the details about the convergence of the improper integral (L'Hospital's rule might come in handy).

$$\mathcal{L}(f(t)) = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt = \int_0^1 t^2 \mathrm{e}^{-st}\, dt + \int_1^\infty (2+t)\mathrm{e}^{-st}\, dt$$

To integrate each of these, we need to integrate by parts:

$+$	t^2	e^{-st}
$-$	$2t$	$-\frac{1}{s}\mathrm{e}^{-st}$
$+$	2	$\frac{1}{s^2}\mathrm{e}^{-st}$
$-$	0	$-\frac{1}{s^3}\mathrm{e}^{-st}$

$+$	$2+t$	e^{-st}
$-$	1	$-\frac{1}{s}\mathrm{e}^{-st}$
$+$	0	$\frac{1}{s^2}\mathrm{e}^{-st}$

For the first integral,

$$-\mathrm{e}^{-st}\left(\frac{t^2}{s} + \frac{2t}{s^2} + \frac{2}{s^3}\right)\Bigg|_0^1 = -\mathrm{e}^{-s}\left(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}\right) + \frac{2}{s^3}$$

For the second integral,

$$-\mathrm{e}^{-st}\left(\frac{2+t}{s} + \frac{1}{s^2}\right)\Bigg|_1^\infty = \lim_{t\to\infty}\left[-\mathrm{e}^{-st}\left(\frac{2+t}{s} + \frac{1}{s^2}\right)\right] + \mathrm{e}^{-s}\left(\frac{3}{s} + \frac{1}{s^2}\right)$$

For this limit, we see that, by L'Hospital's rule:

$$\lim_{t\to\infty}\left[-\mathrm{e}^{-st}\left(\frac{2+t}{s} + \frac{1}{s^2}\right)\right] = -\lim_{t\to\infty}\frac{\frac{2+t}{s} + \frac{1}{s^2}}{\mathrm{e}^{st}} = \lim_{t\to\infty}\frac{\frac{1}{s}}{s\mathrm{e}^{st}} = 0, \quad s > 0$$

Overall, the solution is:

$$-\mathrm{e}^{-s}\left(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}\right) + \frac{2}{s^3} + \mathrm{e}^{-s}\left(\frac{3}{s} + \frac{1}{s^2}\right)$$

Note: You could check your answer by rewriting $f(t)$ in terms of the Heaviside function, then use the table (not necessary, but shown here for completeness). If $t \ge 0$,

$$f(t) = (1 - u(t-1))t^2 + u(t-1)(2+t) = t^2 - u_1(t)t^2 + u_1(t)(2+t)$$

For the term:

$$u_1(t)t^2 \quad \Rightarrow \quad f(t-1) = t^2 \quad \Rightarrow \quad f(t) = (t+1)^2 = t^2 + 2t + 1$$

So the Laplace transform is:

$$\mathcal{L}(u_1(t)t^2) = \mathrm{e}^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

For the third term,

$$u_1(t)(2+t) \quad \Rightarrow \quad f(t-1) = 2+t \quad \Rightarrow \quad f(t) = 2+(t+1) = 3+t$$

so the Laplace transform is:

$$\mathcal{L}(u_1(t)(2+t) = \mathrm{e}^{-s}\left(\frac{3}{s} + \frac{1}{s^2}\right)$$

2. Problem 14, page 313. You may either use the formula given in the text, or the formula:

$$\mathrm{e}^{(a+ib)t} = \mathrm{e}^{at}\cos(bt) + i\mathrm{e}^{at}\sin(bt)$$

and you may use Table Entry 2. Of course, you may NOT use Table Entry 10.

Given Euler's formula,

$$\mathrm{e}^{(a+ib)t} = \mathrm{e}^{at}\cos(bt) + i\mathrm{e}^{at}\sin(bt)$$

The desired Laplace transform can be found via:

$$\mathcal{L}(\mathrm{e}^{(a+ib)t}) = \mathcal{L}(\mathrm{e}^{at}\cos(bt)) + i\mathcal{L}(\mathrm{e}^{at}\sin(bt))$$

Using Table Entry 2,

$$\frac{1}{s-(a+ib)} = \mathcal{L}(\mathrm{e}^{at}\cos(bt)) + i\mathcal{L}(\mathrm{e}^{at}\sin(bt))$$

$$\frac{1}{(s-a)-ib} = \mathcal{L}(\mathrm{e}^{at}\cos(bt)) + i\mathcal{L}(\mathrm{e}^{at}\sin(bt))$$

$$\frac{(s-a)+i\,b}{(s-a)^2+b^2} = \mathcal{L}(\mathrm{e}^{at}\cos(bt)) + i\mathcal{L}(\mathrm{e}^{at}\sin(bt))$$

$$\frac{(s-a)}{(s-a)^2+b^2} + i\frac{b}{(s-a)^2+b^2} = \mathcal{L}(\mathrm{e}^{at}\cos(bt)) + i\mathcal{L}(\mathrm{e}^{at}\sin(bt))$$

Therefore, the Laplace transform of $\mathrm{e}^{at}\cos(bt)$ is

$$\frac{(s-a)}{(s-a)^2+b^2}$$

3. Find, using the table: $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+2s-3}\right)$

We first perform Partial Fraction Decomposition:

$$\frac{s}{s^2+2s-3} = \frac{A}{s+3} + \frac{B}{s-1} = \frac{3}{4}\,\frac{1}{s+3} + \frac{1}{4}\,\frac{1}{s-1}$$

so that the Laplace transform is:

$$\frac{3}{4}\mathrm{e}^{-3t} + \frac{1}{4}\mathrm{e}^{t}$$

4. Find an expression for $Y(s)$, do not solve for $y(t)$:

$$y'' - 4y' + 4y = \mathrm{e}^t \cos(t), \quad y(0) = 0, y'(0) = 1$$

Taking the Laplace transform of both sides,

$$(s^2 Y - 0 - 1) - 4(sY - 0) + 4Y = \frac{s-1}{(s-1)^2 + 1}$$

$$Y(s) = \frac{s-1}{((s-1)^2+1)(s^2-4s+4)} + \frac{1}{s^2-4s+4}$$

5. Solve the following IVP using the method of Laplace Transforms:

$$y'' + 3y' + 2y = \mathrm{e}^{-t} \qquad y(0) = 1 \quad y'(0) = 0$$

$$s^2 Y - s + 3(sY - 1) + 2Y = \frac{1}{s+1} \quad \Rightarrow \quad (s^2 + 3s + 2)Y = \frac{1}{s+1} + s + 3$$

Therefore,

$$Y = \frac{1}{(s+1)(s^2+3s+2)} + \frac{s+3}{s^2+3s+2}$$

We could keep these separate, or combine them (the following solution will combine them). Factor the denominator:

$$Y = \frac{1}{(s+1)^2(s+2)} + \frac{s+3}{(s+1)(s+2)} = \frac{s^2+4s+4}{(s+1)^2(s+2)} = \frac{s+2}{(s+1)^2}$$

Use partial fractions, or see that:

$$Y = \frac{s+1+1}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

so that $y(t) = \mathrm{e}^{-t} + t\mathrm{e}^{-t}$.