STUDY GUIDE: Ch. 1-2

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This review is organized into four main areas: Theory, Analysis, Methods and Models.

We've seen that a differential equation defines a family of functions (an initial value problem defines a specific function). As such, ODEs provide a powerful tool for modeling.

When we solve an ODE, we not only want to get an analytic solution, but we also want to understand graphical analysis (direction fields, phase plots) and we want to be able to analyze the solution- Is the solution unique? What is its behavior over the long term?, etc.

1 Existence and Uniqueness Theorem

1. Linear: $y' + p(t)y = g(t)$ at (t_0, y_0):

   If p, g are continuous on an interval I that contains t_0, then there exists a unique solution to the initial value problem and that solution is valid for all t in the interval I.

2. General Case: $y' = f(t, y)$, (t_0, y_0):

   (a) If f is continuous on a small rectangle containing (t_0, y_0), then there exists a solution to the initial value problem.

   (b) If $\partial f/\partial y$ is continuous on that small rectangle containing (t_0, y_0), then that solution is unique.

   (c) We can only guarantee that the solution persists on a small interval about (t_0, y_0). To find the full interval, we need to actually solve the initial value problem.

2 Analysis of Solutions

1. Construct a direction field: Since $y' = f(t, y)$, at each value of (t, y), we can compute the local slope, $y'(t)$. Isoclines can be used to help: An isocline is determined by setting the derivative equal to a constant k and plotting the curve determined by: $k = f(t, y)$.

2. The Phase Diagram, and the Direction Field: Given that $y' = f(y)$, we can plot y' vs. y. This gives us information that we can translate to the direction field, a plot of y vs t. This information is summarized in the table below.

In Phase Diagram:	In Direction Field:
y intercepts	Equilibrium Solutions
$+$ to $-$ crossing	Stable Equilibrium
$-$ to $+$ crossing	Unstable Equilibrium
$y' > 0$	y increasing
$y' < 0$	y decreasing
y' and df/dy same sign	y is concave up
y' and df/dy mixed	y is concave down

3 Methods

The primary way of solving first order ODEs is to first classify it by type, however, recall that a differential equation may satisfy multiple classifications. For example, $y' = ay + b$ is linear, separable, exact and autonomous.

- Linear: $y' + p(t)y = g(t)$.

  Use the integrating factor: $e^{\int p(t)\,dt}$

  Algebra: Recall that $e^{a+b} = e^a e^b$, $e^{\ln(A)} = A$

- Separation of variables: $y' = f(y)g(t)$

  Separate variables: $(1/f(y))\,dy = g(t)\,dt$

- Exact: $M(x,y) + N(x,y)\frac{dy}{dx}$, where $N_x = M_y$.

  Solution: Set $f_x(x,y) = M(x,y)$. Integrate w/r to x. Check that $f_y = N(x,y)$, and add a function of y if necessary.

4 Models

1. Construct an autonomous differential equation to model population growth in the standard model and with an environmental carrying capacity. Be able to solve these models analytically (usually requires partial fractions) and graphically.

2. Be able to construct the differential equation corresponding to the tank mixing problem. Be able to solve it (it will be a linear first order equation) and analyze it (using a phase diagram, if possible).

3. Given Newton's Law of cooling model, be able to find the coefficients in the model, solve it analytically, and analyze the behavior of the solutions.

4. An object falling: $mv' = mg - kv$. For these problems, I will give you the constant(s) for g.

5 Construct A Differential Equation

It is useful to construct your own differential equations so that you get a better feel for how they come about. This can be done through the modeling process, or directly. Be sure to try out the sample problems.

6 Background Material

Like algebra is to Calculus, Calculus is to ODEs. Here are some particular topics to remember:

- Derivative formulas: The usual, but also the inverse trig formulas! It is possible to construct the formula yourself- For example:

  $$y = \sin^{-1}(x) \quad \Rightarrow \quad \sin(y) = x \quad \Rightarrow \quad \cos(y)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

  To write $\cos(y)$ in terms of x, draw a triangle that satisfies $\sin(y) = x$, then determine $\cos(y)$ (which is $\sqrt{1-x^2}$) by the Pythagorean Theorem).

- Integration Techniques: We've talked about integration by parts using a table, and partial fraction decomposition.

- Recall that $f(x,y) = c$ determines y **implicitly** as a function of x. This idea was central to separable equations and exact equations.

- The Fundamental Theorem of Calculus (Part I). If $f(t)$ is continuous on the interval $[a,b]$, then the function defined by:

  $$g(t) = \int_0^t f(x)\,dx$$

  exists, is continuous on $[a,b]$ and differentiable on (a,b). In fact, $g'(t) = f(t)$.

  This was used explicitly in a couple of places- In the Existence and Uniqueness Theorem for linear first order ODEs, and in Problem 14, p. 25 (Sect 1.3).