Homework Solutions
(Replaces Section 3.9)

1. Find the general solution of the given differential equation:

$$y'' + 3y' + 2y = \cos(t)$$

First, get the homogeneous part of the solution by solving the characteristic equation:

$$r^2 + 3r + 2 = 0 \quad \Rightarrow \quad (r+2)(r+1) = 0 \quad \Rightarrow \quad r = -1, -2$$

Therefore, $y_h(t) = C_1 e^{-t} + C_2 e^{-2t}$. You could solve for the particular solution one of two ways. For extra practice, you might try both. Here they are:

- Method of Undetermined Coefficients

$$y_p(t) = A\cos(t) + B\sin(t) \quad y_p' = B\cos(t) - A\sin(t) \quad y_p'' = -A\cos(t) - B\sin(t)$$

so that, looking at the coefficients of $\cos(t)$ and $\sin(t)$, we have:

$$(-A + 3B + 2A)\cos(t) + (-B - 3A + 2B)\sin(t) = \cos(t)$$

Therefore, by Cramer's Rule:

$$\begin{array}{rl} A + 3B & = 1 \\ -3A + B & = 0 \end{array} \quad \Rightarrow \quad A = \frac{1}{10} \quad B = \frac{3}{10}$$

The solution is $C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$

2. Pictured below are the graphs of several solutions to the differential equation:

$$y'' + by' + cy = \cos(\omega t)$$

Match the figure to the choice of parameters.

$$r^2 + br + c = 0 \quad \Rightarrow \quad \frac{1}{2}$$

Choice	b	c	ω
(A)	5	3	1
(B)	1	3	1
(C)	5	1	3
(D)	1	1	3

We will discuss each situation before locating the curve on the graph.

(a) With $b = 5, c = 3$, we have $r = -0.69$ and -4.3. Therefore, our overall solution will quickly die off to just the particular solution, which will have a period of 2π.

(b) With $b = 1$, $c = 3$, we have $r = -0.5 \pm 1.65i$. Initially, we will have a combination of sinusoidals, but it again will die off (not as quickly as before) to a function whose period is 2π.

(c) With $b = 5, c = 1$, we have something similar to (a), but now the forcing function has period $2\pi/3 \approx 2.09$.

(d) The pseudo-(natural) frequency of the homogeneous part of the equation is $\sqrt{3}2$, but this again dies off leaving a function that is periodic with period about 2.09.

Now, graphs (ii) and (iii) will correspond to (a) and (b) (by the periods). It looks like (iii) will probably correspond to (a), because of how rapidly the solution converges to the particular solution.

So far: (ii) corresponds to (b) and (iii) corresponds to (a).

It would be a safe guess to bet that (i) corresponds to (d) and that leaves (iv) corresponding to (c).

3. Recall that

$$\text{Real}(e^{i\theta}) = \cos(\theta) \qquad \text{Imag}(e^{i\theta}) = \sin(\theta)$$

Show that, given the DE below we can use the ansatz $y_p = Ae^{3ti}$ (the real part),

$$y'' + 4y = 2\cos(3t)$$

and we will get the particular solution,

$$A = -\frac{2}{5} \qquad \Rightarrow \qquad y_p(t) = -\frac{2}{5}\cos(3t)$$

Solution: First differentiate, then substitute into the DE:

$$y_p(t) = Ae^{3it} \quad y_p' = 3iAe^{3it} \quad y_p'' = 9i^2Ae^{3it} = -9Ae^{3it}$$

We notice that $2\cos(3t)$ is the real part of $2e^{3it}$, so:

$$-9Ae^{3it} + 4Ae^{3it} = 2e^{3it} \quad \Rightarrow \quad -5A = 2 \quad \Rightarrow \quad A = -\frac{2}{5}$$

Therefore, taking the real part of $-\frac{2}{5}e^{3it}$ gives us our particular solution.

4. Fill in the question marks with the correct expression:

Given the undamped second order differential equation, $y'' + \omega_0^2 y = A\cos(\omega t)$, we see "beating" if ($|\omega - \omega_0|$ is small) In particular, the longer period wave has a period that gets **longer** as $\omega \to \omega_0$, and its amplitude gets **bigger**

5. Find the solution to $y'' + 9y = 2\cos(3t)$, $y(0) = 0$, $y'(0) = 0$ by first solving the more general equation: $y'' + 9y = 2\cos(at)$, $y(0) = 0$, $y'(0) = 0$, then take the limit of your solution as $a \to 3$.

Solution: This is going through the steps of the argument that we also did in class. First, the homogeneous part of the solution is:

$$y_h(t) = C_1\cos(3t) + C_2\sin(3t)$$

The particular part is (differentiate to substitute into the DE):

$$y_p = A\cos(at) + B\sin(at) \quad y_p' = Ba\cos(at) - Aa\sin(at) \quad y_p'' = -Aa^2\cos(at) - Ba^2\sin(at)$$

And,

$$y_p'' + 3y_p = A\left(-a^2+9\right)\cos(at) + B\left(-a^2+9\right) = 2\cos(at) \quad \Rightarrow \quad A = \frac{2}{9-a^2} \quad B = 0$$

The particular part of the solution is:

$$y_p(t) = \frac{2}{9-a^2}\cos(at)$$

Put everything together to solve with the initial conditions, $y(0) = 0$ and $y'(0) = 0$:

$$y = C_1\cos(3t) + C_2\sin(3t) + \frac{2}{9-a^2}\cos(at) \quad \Rightarrow \quad 0 = C_1 + \frac{2}{9-a^2} \quad \Rightarrow \quad C_1 = -\frac{2}{9-a^2}$$

and, using $y'(0) = 0$,

$$0 = 0 + 3C_2 + 0 \quad \Rightarrow \quad C_2 = 0$$

Therefore, the overall solution to the IVP is:

$$y(t) = \frac{2}{9-a^2}\left(\cos(at) - \cos(3t)\right)$$

Take the limit as $a \to 3$ using L'Hospital's rule:

$$\lim_{a\to 3}\frac{2(\cos(at)-\cos(3t))}{9-a^2} = \lim_{a\to 3}\frac{-2t\sin(at)}{-2a} = t\sin(3t)$$

6. Suppose a unit mass is attached to a spring, with spring constant $k = 16$. Assuming that damping is negligible ($\gamma = 0$), suppose that we lightly tap the mass with a hammer (downward) every T seconds.

   Suppose the first tap is at $t = 0$, and before that time the mass is at rest[1]. Describe what you think will happen to the motion of the mass for the following choices of the tapping period T: (a) $T = \pi/2$ (b) $T = \pi/4$

   Note that the homogeneous solution is

   $$y_h(t) = C_1\cos(4t) + C_2\sin(4t) = R\cos(4t - \delta)$$

   so that the period of the homogeneous solution is

   $$\frac{2\pi}{4} = \frac{\pi}{2}$$

   Using a period of $pi/2$ for the hammer strikes will then result in a phenomena much like resonance, where the amplitude of the solution will begin to go to infinity.

   Using a period of $\pi/4$, it is possible that the motion of the spring stopped, since the hammer strike is moving in the opposite direction of the motion. It would then start up again in another $\pi/4$ units of time, and repeat.

7. (Extra Practice) Can the following functions be linearly independent solutions to a second order linear homogeneous differential equation? Why or why not?

   The two functions can generally NOT be linearly independent solutions. We see that the Wronskian is zero at approximately $t = 2\pi$ and $t = 4\pi$ (the functions share a $t-$intercept).

[1]If you want to algebraically describe this, use initial conditions $u(0) = 0$ and $u'(0) = 1$. We will solve this problem completely after Spring Break.