Lecture Notes: To Replace 7.1/7.2

Summary:

- What is a system of equations?
- Definition of a matrix, vector, matrix-vector multiplication, matrix-matrix multiplication, determinant of a matrix, transpose.
- Special matrices: I and A^{-1}.
- Parametric equations, derivatives, integrals
- Convert an n^{th} order DE to an equivalent system of first order.
- Solution to 2×2 systems of first order.

Motivating example: System of two tanks.

Matrices and Operations on Matrices

A system of 2 equations in 2 unknowns can be converted in a compact way to a matrix-vector equation:

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

A **matrix** is simply an array of numbers, and the size of a matrix is defined as the number of *rows* $\times$ the number of *columns*. We will work with 2×2 matrices. A **vector** is a column of numbers. We say that a vector belongs to $\mathbb{R}^n$ if it has n real numbers as its components. The definition above gives meaning to matrix-vector multiplication.

Example: Compute the following:

$$\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Matrix-Matrix multiplication is defined via matrix-vector multiplication. Think of the second matrix in terms of its columns:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

For what sizes of matrices is matrix-matrix multiplication defined?

Special Case: Scalar multiplication (See example)

Example: Compute the following:

$$\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -3+0 & -1+0 \\ 3+2 & 1-4 \end{bmatrix}$$

$$5 \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 5 & -10 \end{bmatrix}$$

The determinant, the transpose and the trace.

In the transpose, the old columns make up the new rows (diagonal elements are left unchanged).

$$\det(A) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad \mathrm{Tr}(A) = a + d$$

We have seen this determinant when we use Cramer's Rule to solve a system of equations. We'll see it again momentarily.

Inverses and the Identity

There are two special matrices used in matrix multiplication: The identity and the inverse. The identity matrix is a matrix whose only non-zero elements are the ones along its diagonal. It can be any square size, as needed (use the one for which the given multiplication is defined).

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

You will verify in the exercises that, for any matrix A, the identity works like the number 1 in the real numbers:

$$AI = IA = A$$

The inverse of a matrix A is another matrix, A^{-1} so that:

$$AA^{-1} = A^{-1}A = I$$

You will verify in the exercises that, given a 2×2 matrix, the inverse can be written down directly:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{1}$$

Solving the System

In solving a system of equations, there are three (and only three) possible outcomes: (i) Exactly one solution (intersecting lines), (ii) No Solution (parallel lines), (iii) an infinite number of solutions (the same line).

Theorem If the matrix of coefficients has an inverse, then the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, $\mathbf{x} = A^{-1}\mathbf{b}$ (which could also be found by Cramer's Rule or computing the inverse directly using Equation 1).

Corollary 1: If the matrix of coefficients has a non-zero determinant, then there is exactly one solution to the system of equations (because we can compute the inverse).

Corollary 2: If we are solving $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$, then we obtain an infinite number of solutions *only* when $\det(A) = 0$ (You might notice that in this system, there are only two possible outcomes rather than three. What are they?)

Examples:

1. Solve the system:

$$\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

SOLUTION: The determinant is -2, so there is exactly one solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1/2 \end{bmatrix}$$

We can verify that this is a solution:

$$\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

2. Solve the system:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

SOLUTION: The determinant is 0, so there is an infinite number of solutions (NOTE: We cannot have "no solution", because $x = 0$ and $y = 0$ is a "trivial" solution). To represent the infinite number of choices, go back to the original equations:

$$\begin{aligned} x + 2y &= 0 \\ 2x + 4y &= 0 \end{aligned}$$

The second line is a constant multiple of the first. Therefore, for these equations to be true must mean that either

$$x = -2y \qquad \text{or } y = -\frac{1}{2}x$$

So, for example, any of these choices of vectors would solve the system:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \text{ or } \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ldots$$

Generically, any vector of the following form would solve the system:

$$\begin{bmatrix} -2c \\ c \end{bmatrix} = c \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Parametric Equations

You've seen parametric equations in Calculus- They are a mapping from the real line to $\mathbb{R}^n$. Typically, we will only use the plane ($\mathbb{R}^2$), so our functions will look like the following, where differentiation and integration are defined elementwise:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} \quad \int \mathbf{x}(t)\, dt = \begin{bmatrix} \int x_1(t)\, dt \\ \int x_2(t)\, dt \end{bmatrix}$$

When we visualize these in the plane, we will see a curve, where each point of the curve is defined as $(x_1(t), x_2(t))$. Notice that you could also visualize these as two separate curves, $(t, x_1(t))$ and $(t, x_2(t))$.

Systems of DEs and Parametric Equations

Definition: An **autonomous** system of first order linear differential equations is a system of the form:

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x}' = A\mathbf{x}$$

Definition: A **solution** to the system is a parametric function that satisfies the given relationship.
Definition: The **trivial solution**: the origin is always a solution to the autonomous linear system. In fact, any constant solution to $A\mathbf{x} = \mathbf{0}$ is an **equilibrium solution.**

Examples

1. Show that $\mathbf{x}(t) = [\cos(t), \sin(t)]^T$ solves the system:

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

SOLUTION: We see that $x_1'(t) = -\sin(t)$ and $x_2'(t) = \cos(t)$. Therefore, $x_1' = -x_2$ and $x_2' = x_1$. This is the system of differential equations represented by the matrix-vector equation given. We also note that the solution to the system would be plotted as a circle in the plane.

2. Find the equilibrium solutions to:

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{x}$$

We set the system to zero and solve (See the previous section). The solutions are:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c \begin{bmatrix} -2 \\ 1 \end{bmatrix} =$$

There is a line of equilibrium solutions (can you write the equation of the line in the plane?)

3. Verify that the following function solves the following system:

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \qquad \mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mathrm{e}^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \mathrm{e}^{-t}$$

We will substitute this $\mathbf{x}$ into the system to see if we get a true statement (just like we did way back in Chapter 1):

$$\mathbf{x}' = 3c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mathrm{e}^{3t} - c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \mathrm{e}^{-t} = \begin{bmatrix} 3c_1\mathrm{e}^{3t} - c_2\mathrm{e}^{-t} \\ 6c_1\mathrm{e}^{3t} + 2c_2\mathrm{e}^{-t} \end{bmatrix}$$

And, compare this to:

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} c_1\mathrm{e}^{3t} + c_2\mathrm{e}^{-t} \\ 2c_1\mathrm{e}^{3t} - 2c_2\mathrm{e}^{-t} \end{bmatrix} = \begin{bmatrix} 3c_1\mathrm{e}^{3t} - c_2\mathrm{e}^{-t} \\ 6c_1\mathrm{e}^{3t} + 2c_2\mathrm{e}^{-t} \end{bmatrix}$$

Second order DEs to First Order Systems

We can convert every second order ODE into an equivalent system of first order equations. This is important in two aspects- (i) Most computer software systems require you to do this, and (ii) The systems are more general.

The idea is to make a substitution:

$$ay'' + by' + cy = 0 \qquad \text{Let} \quad \begin{array}{ll} x_1 & = y \\ x_2 & = y' \end{array} \qquad \Rightarrow \qquad \begin{array}{ll} x_1' & = y' \\ x_2' & = y'' = -(b/a)y' - (c/a)y \end{array}$$

Writing this in terms of x_1, x_2, we get:

$$ay'' + by' + cy = 0 \quad \Rightarrow \quad \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -(c/a) & -(b/a) \end{bmatrix} \mathbf{x}$$

In terms of our models, notice that we are now solving for both position and velocity at the same time, rather than just position.

One question that you might have: Does every (autonomous) system correspond to a second order system? The answer is: Many do.

Systems to Second Order

Rather than giving a general formula, which is somewhat long and not terribly meaningful, let's try a specific problem. We will try to convert the following into a second order DE:

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$$

The coefficients in the first equation will be easier to use for the substitution: Solve it for x_2 (and therefore also x_2'), and use that substitution into the second equation. The result is a second order DE in x_1:

$$\begin{aligned} x_2 &= x_1' - x_1 \\ x_2' &= x_1'' - x_1' \end{aligned} \qquad \text{2d equation becomes} \qquad x_1'' - x_1' = 4x_1 + (x_1' - x_1) \quad \Rightarrow$$

$$x_1'' - 2x_1' - 3x_1 = 0$$

Notice that we could have used the second equation to solve for x_1, then substitute (and its derivative) into the first. Just for fun, let's try it and see what happens:

$$\begin{aligned} x_1 &= \tfrac{1}{4}x_2' - \tfrac{1}{4}x_2 \\ x_1' &= \tfrac{1}{4}x_2'' - \tfrac{1}{4}x_2' \end{aligned} \qquad \text{substitution} \qquad \frac{1}{4}x_2'' - \frac{1}{4}x_2' = \frac{1}{4}x_2' - \frac{1}{4}x_2 + x_2$$

Clear fractions by multiplying by 4, then simplify to get: The same equation as before!

From Chapter 3, we can solve the system- But be careful, we need to be consistent. We will solve the system both ways to show you why.

The characteristic equation for the DE has roots $r = 3, -1$, so the general solution in the first equation is:

$$x_1 = c_1 e^{3t} + c_2 e^{-t}$$

Then, using the given substitution, $x_2 = x_1' - x_1$, so in this case,

$$x_2 = 3c_1 e^{3t} - c_2 e^{-t} - c_1 e^{3t} - c_2 e^{-t} = 2c_1 e^{3t} - 2c_2 e^{-t}$$

This is the solution we obtained in Example 3 of the previous section. However, if we had used the other substitution:

$$x_2 = k_1 e^{3t} + k_2 e^{-t}$$

then $x_1 = \frac{1}{4}(x_2' - x_2)$. To finish it up,

$$x_1 = \frac{1}{2}k_1 e^{3t} - \frac{1}{2}k_2 e^{-t}$$

To make the two solutions, we notice that $k_1 = 2c_1$ and $k_2 = 2c_2$. We do indeed get the correct solution either way- as long as we are consistent!