Homework (M 244): To Replace 7.1/7.2

1. Exercise 22, p 363 (Section 7.1)

2. Short Answer:

   (a) Is every second order linear homogeneous differential equation (with constant coefficients) equivalent to a system of first order equations?

   (b) Can every 2×2 system of DEs be converted into an equivalent second order system? (Hint: To do our technique, what must be true?)

3. Give the solution to each system. If it has an infinite number of solutions, give your answer in vector form:

$$\begin{aligned} 3x + 2y &= 1 \\ 2x - y &= 3 \end{aligned} \qquad \begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= 3 \end{aligned} \qquad \begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= 2 \end{aligned}$$

4. Write each of the previous systems in matrix-vector form. Verify that the determinant of the first matrix is not zero, but is zero for the second and third.

5. Write each system of differential equations in matrix-vector form or write the system from the matrix-vector form:

$$\begin{aligned} x_1' &= 3x_1 - x_2 \\ x_2' &= 9x_1 - 3x_2 \end{aligned} \qquad \mathbf{x}' = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

6. Find the equilibrium solutions to the previous autonomous linear differential equations.

For each of the problems below, define

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

and I is the identity matrix:

7. Compute $(B - 4I)\boldsymbol{b}$
8. Compute $\det(B - 4I)$
9. Are AB and BA the same?
10. Compute $A^{-1}\boldsymbol{b}$
11. Verify: $A\boldsymbol{b} - 3\boldsymbol{b} = (A - 3I)\boldsymbol{b}$
12. Compute $B^T B$, $\mathrm{Tr}(A)$, and $\mathrm{Tr}(B)$

13. If $\mathbf{x}$ is as defined below, compute $\mathbf{x}'(t)$, and $\int_0^1 \mathbf{x}(t)\, dt$:

$$\mathbf{x}(t) = \begin{bmatrix} t^2 - 3 \\ 3e^t - 2e^{3t} \end{bmatrix}$$

14. Solve the system of equations given by first converting it into a second order linear ODE (then use Chapter 3 methods):

(a) $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ (b) $\mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{x}$

15. The four graphs in the attached figure show the direction fields for the four systems of differential equations below. Try to reason out which direction field goes with which system.

(a) $\mathbf{x} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \mathbf{x}$ (c) $\mathbf{x} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{x}$

(b) $\mathbf{x} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ (d) $\mathbf{x} = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$

16. Convert the following second order differential equations into a system of autonomous, first order equations. Using methods from Chapter 3, give the solution to the system. An example follows before the exercises:

$$y'' + 3y' + 2y = 0$$

SOLUTION: We'll get the homogenous solution first. The roots to the characteristic equation are $-1, -2$. The general solution is:

$$y = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-2t}$$

To get an equivalent system, let $x_1 = y$ and $x_2 = y'$. Then

$$x_1' = y' = x_2 \qquad x_2' = y'' = -2y - 3y' = -2x_1 - 3x_2$$

so the system is (in matrix-vector form):

$$\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}$$

Since $x_1 = y$, then $x_1 = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-2t}$. Since $x_2 = y'$, then $x_2 = -C_1 \mathrm{e}^{-t} - 2C_2 \mathrm{e}^{-2t}$. In vector form, this means our solution is:

$$\mathbf{x} = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \mathrm{e}^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Here we go:

(a) $y'' + 4y' + 3y = 0$ (c) $y'' + 4y = 0$

(b) $y'' + 5y' = 0$ (d) $y'' - 2y' + y = 0$

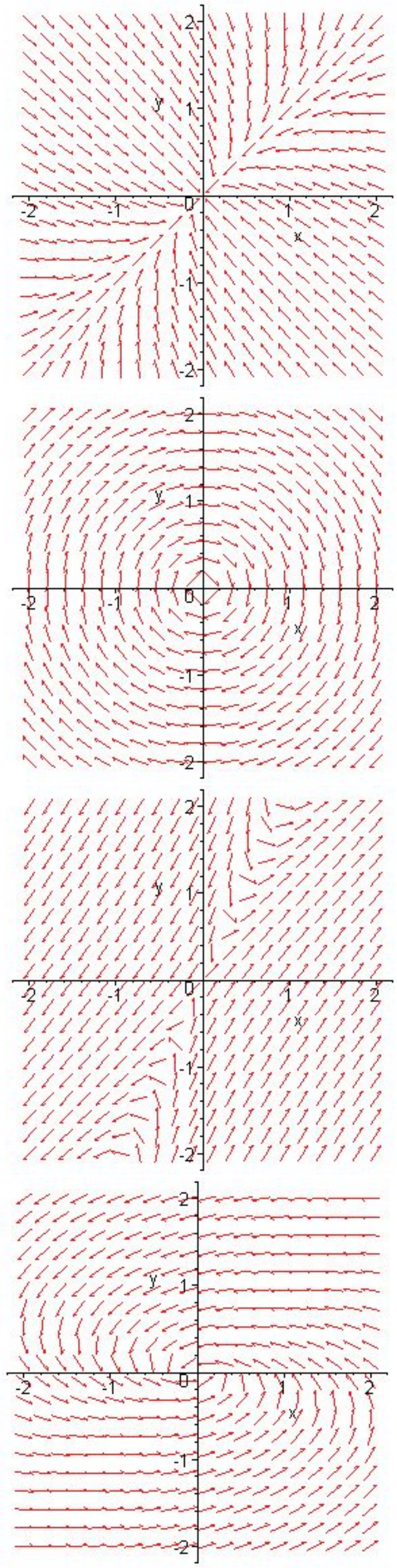

Figure 1: Figure for "Match the system with the Phase plane" problem.