Lecture Notes (Day 2-3): To Replace Chap 7

Starting with the autonomous linear system,

$$\mathbf{x}' = A\mathbf{x}$$

We saw a hint at how we might solve the general system using the exponential function. We will use as an ansatz:

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$$

For this to be a solution, we must have that:

$$\mathbf{x}' = \lambda \mathrm{e}^{\lambda t}\mathbf{v} = A\mathbf{x} = A\mathrm{e}^{\lambda t}\mathbf{v}$$

Or, the constant λ and the vector $\mathbf{v}$ must satisfy the relationship:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Such a constant-vector pair are called an eigenvalue and eigenvector for the matrix A.

Example: Show that $\lambda = 2$, $\mathbf{v} = [-1,\ 1]^T$ is an eigenvalue/vector for the matrix

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

SOLUTION:

$$B\mathbf{v} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda\mathbf{v}$$

Computing Eigenvalues and Eigenvectors:

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Rightarrow \quad \begin{array}{ll} av_1 + bv_2 & = \lambda v_1 \\ cv_1 + dv_2 & = \lambda v_2 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} (a-\lambda)v_1 & +bv_2 & = 0 \\ cv_1 & +(d-\lambda)v_2 & = 0 \end{array}$$

That is the key system of equations. We saw last time $A\mathbf{x} = 0$ has exactly the zero solution iff $\det(A) \neq 0$. Therefore, for there to be a non-zero eigenvector, we must have that:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Theorem: The eigenvalues for the 2×2 matrix A are found by solving the **characteristic equation**:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

So, given A, compute the $\mathrm{Tr}(A)$, the $\det(A)$ and the discriminant,

$$\Delta = (\mathrm{Tr}(A))^2 - 4\det(A)$$

Then the eigenvalues are:

$$\lambda = \frac{Tr(A) \pm \sqrt{\Delta}}{2}$$

Just as in Chapter 3, the form of the solution will depend on whether Δ is positive (two real λ), negative (two complex λ) or zero (one real λ).

Case 1: Real eigenvalues, Two eigenvectors

Example: Solve and analyze the graph of the solutions to:

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x} \qquad \text{Tr}(A) = 1 \quad \det(A) = -2 \quad \Delta = 9$$

The eigenvalues are $\lambda = -1, 2$. The corresponding eigenvectors are found by solving the system above. For $\lambda = -1$:

$$\begin{aligned} (3+1)v_1 - 2v_2 &= 0 \\ 2v_1 + (-2+1)v_2 &= 0 \end{aligned} \qquad 2v_1 - v_2 = 0 \qquad v_2 = 2v_1 \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = 2$:

$$\begin{aligned} (3-2)v_1 - 2v_2 &= 0 \\ 2v_1 + (-2-2)v_2 &= 0 \end{aligned} \qquad v_1 - 2v_2 = 0 \qquad v_1 = 2v_2 \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(Exercise is to verify the following) The solution to the system of differential equations is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

To sketch the graph, notice that, if $c_1 = 0$, then the solution starts on a multiple of $[2,1]^T$ and moves along the vector to ∞- Same for negative multiples. If $c_2 = 0$, then the solution tends to zero along the vector $[1,2]^T$. The equilibrium solution (the origin) in this instance is called a **saddle point**. That is, some most solutions tend towards infinity, except for some special cases.

Case 2: Complex Eigenvalues

Suppose we have a complex eigenvalue, $\lambda = a \pm ib$. Use one of them to construct the corresponding eigenvector (complex) $\mathbf{v}$.

Theorem: Given $\lambda = a + ib$, $\mathbf{v}$, the solution to the system of differential equations is:

$$\mathbf{x}(t) = C_1 \text{Real}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \text{Imag}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

Notice that this is the extension of what we did in Chapter 3 (verify in the exercises).

Example: $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ The trace is zero, the determinant is 1, the discriminant is -4. Therefore, $\lambda = \pm i$. Solve for one of the eigenvectors. Using $\lambda = i$, we have:

$$\begin{aligned} (2-i)v_1 - 5v_2 &= 0 \\ 1v_1 + (-2-i)v_2 &= 0 \end{aligned}$$

Using the second equation, $v_1 = (2+i)v_2$, and we have our eigenvalue/eigenvector pair. Now we compute the needed quantity, $e^{\lambda t}\mathbf{v}$:

$$e^{it}\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = (\cos(t) + i\sin(t))\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} (\cos(t)+i\sin(t))(2+i) \\ \cos(t)+i\sin(t) \end{bmatrix}$$

Simplifying, we get:

$$\begin{bmatrix} (2\cos(t) - \sin(t)) + i(2\sin(t) + \cos(t)) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

The solution is:

$$\mathbf{x}(t) = C_1\begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_2\begin{bmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}$$

Case 3: One Real Eigenvalue, One Eigenvector

In the rare occurrence that you have one eigenvalue but two eigenvectors (we'll do this in class), go to Case 1. Otherwise, we have the more general case here.

You can read pages 423-424 for more information on this one. This is a special case where we need to find a second eigenvector (called a generalized eigenvector):

- Given an eigenvalue λ and eigenvector $\mathbf{v}$, find the "generalized" eigenvector $\mathbf{w}$ by solving the system:

$$\begin{aligned} (a-\lambda)w_1 + bw_2 &= v_1 \\ c\,w_1 + (d-\lambda)w_2 &= v_2 \end{aligned}$$

The solution to the differential equation is then given by:

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}\left(t\mathbf{v} + \mathbf{w}\right)$$

Of course, in this instance we can always use the method of Chapter 3 to solve this, but we want to note the form of the solution before we talk about the geometry in Chapter 9.

Example:

$$\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\mathbf{x}$$

The trace is 0 and the determinant is 0. Therefore, $\lambda = 0$ is the only eigenvalue. We now get the eigenvector $\mathbf{v}$:

$$4v_1 - 2v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Now the generalized eigenvector $\mathbf{w}$:

$$\begin{aligned} 4w_1 - 2w_2 &= 2 \\ 8w_1 - 4w_2 &= 4 \end{aligned} \qquad 4w_1 - 2w_2 = 2$$

We take any w_1, w_2 that satisfies this relationship- integer solutions are nice (you can change $\mathbf{v}$ if necessary), and in this case we choose $w_1 = 0$ and $w_2 = -1$.

The solution is (in several forms):

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + c_2 \left(t \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 2c_1 + 2c_2 t \\ (4c_1 - c_2) + 4tc_2 \end{bmatrix}$$

We'll check that this is indeed a solution. First, we compute $\mathbf{x}'$ and show that it is equal to $A\mathbf{x}$:

$$\mathbf{x}' = \begin{bmatrix} 2c_2 \\ 4c_2 \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} 2c_1 + 2c_2 t \\ (4c_1 - c_2) + 4tc_2 \end{bmatrix} = \begin{bmatrix} 0 + 2c_2 + 0t \\ 0 + 4c_2 + 0t \end{bmatrix}$$

Summary

To solve $\mathbf{x} = A\mathbf{x}$, find the trace, determinant and discriminant. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on Δ:

- Real λ_1, λ_2 give two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \mathrm{Real}\left(e^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Imag}\left(e^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector $\mathbf{v}$. Get $\mathbf{w}$ that solves $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then:

$$\mathbf{x}(t) = e^{\lambda t}\left(C_1\mathbf{v} + C_2\left(t\mathbf{v} + \mathbf{w}\right)\right)$$