Notes (Ch 9): Poincare Classification

- Earlier, the autonomous DE was: $y' = f(y)$. Goal: Find and classify the equilibria (we looked at plot of y v. y').
- Goal today: Classify the origin in $\mathbf{x}' = A\mathbf{x}$

Summary from last time

Solve $\mathbf{x} = A\mathbf{x}$:

Find the eigenvalues/eigenvectors (through the trace, det and discriminant)

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

Depends on Δ:

- Real λ_1, λ_2 give two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \mathrm{Real}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Imag}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector $\mathbf{v}$. Get $\mathbf{w}$ that solves $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then:

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(C_1\mathbf{v} + C_2\left(t\mathbf{v} + \mathbf{w}\right)\right)$$

So the type of solution depends on Δ, and in particular, where $\Delta = 0$:

$$\Delta = 0 \quad \Rightarrow \quad 0 = (\mathrm{Tr}(A))^2 - 4\det(A)$$

This is a parabola in the $(\mathrm{Tr}(A), \det(A))$ coordinate system.

This cuts the plane into several regions, which we examine below (HANDOUT: Poincaré Diagram)

Template Examples

Use the Poincaré diagram to locate these. The diagram gives "template" shapes, not the actual shapes (we'll see examples below).

- Two distinct eigenvectors: Saddle, source or sink.
  - Both negative eigenvalues: Sink
  - Both positive eigenvalues: Source
  - Mixed sign: Saddle
- Complex eigenvalues: Centers and Spirals
  - Real(λ) $= 0$ Center (Purely periodic, closed curves)
  - Real(λ) > 0: Spiral Source
  - Real(λ) < 0: Spiral Center
- One eigenvector: We have a degenerate source (positive λ) or sink (negative λ).

  Sometimes hard to distinguish these graphically from a spiral.

Using the Poincaré Diagram

In the examples on the next few pages, if the matrix had a double eigenvalue with only one eigenvector, then the generalized eigenvector is given (along with the regular eigenvector).

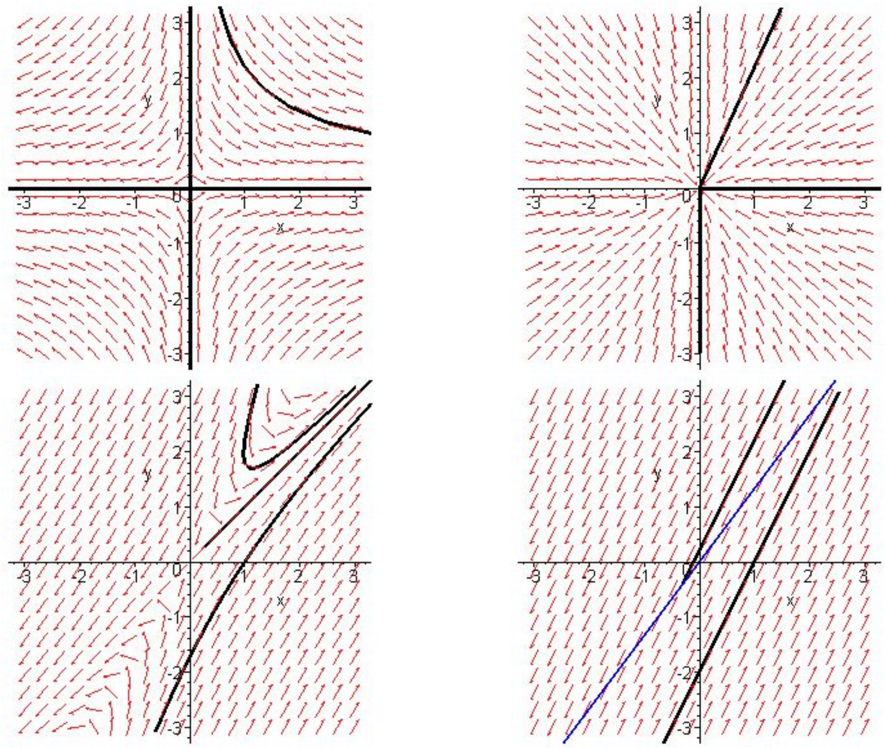

Figure 1: First set of four phase planes. From top to bottom, Saddle, Degenerate Sink, Saddle, Line of Stable fixed points.

First set of examples (See Figure)

System	Tr(A)	det(A)	Δ	Poincare	λ	V
$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$	1	-6	25	Saddle	$3, -2$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	-2	1	0	Degen Sink	$-1, -1$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$	0	-1	4	Saddle	$-1, 1$	$\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$
$\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$	-2	0	4	Line of Stable	$-2, 0$	$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

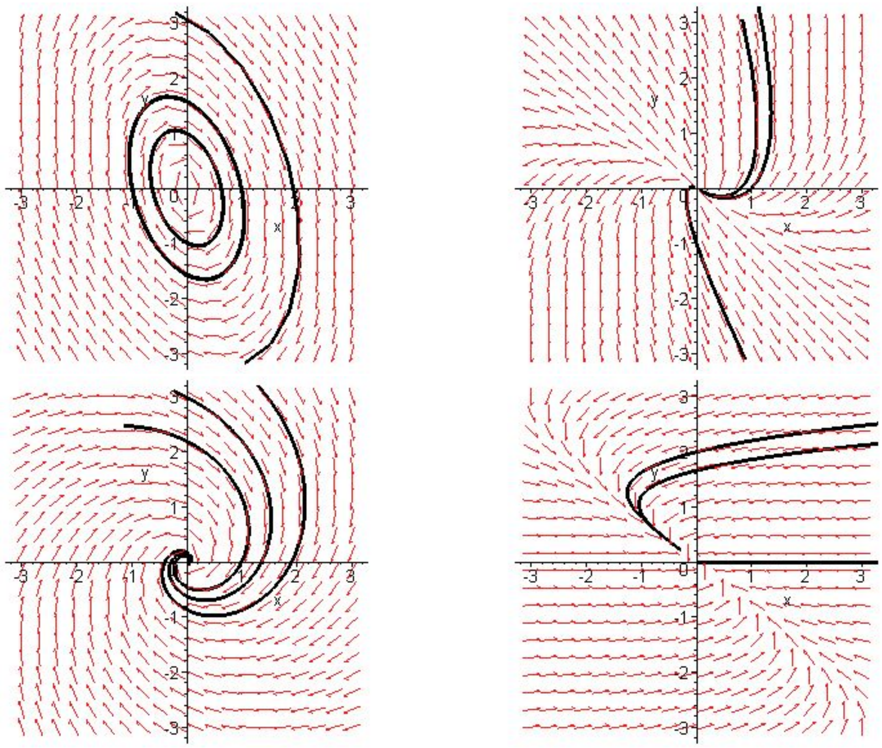

Figure 2: Second set of four phase planes. From top to bottom, Center, Degenerate Source, Spiral Sink, Sink.

System	Tr(A)	det(A)	Δ	Poincare	λ	V
$\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$	0	9	-36	Center	$3i$	$\begin{bmatrix} 2 \\ -1+3i \end{bmatrix}$
$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$	4	4	0	Degen Source	$2, 2$	$\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$
$\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$	-1	$5/4$	-4	Spiral Sink	$-\frac{1}{2}+i$	$\begin{bmatrix} 1 \\ i \end{bmatrix}$
$\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix}$	$-5/4$	$1/4$	$9/16$	Sink	$-1, -1/4$	$\begin{bmatrix} 1 & -4 \\ 0 & 3 \end{bmatrix}$

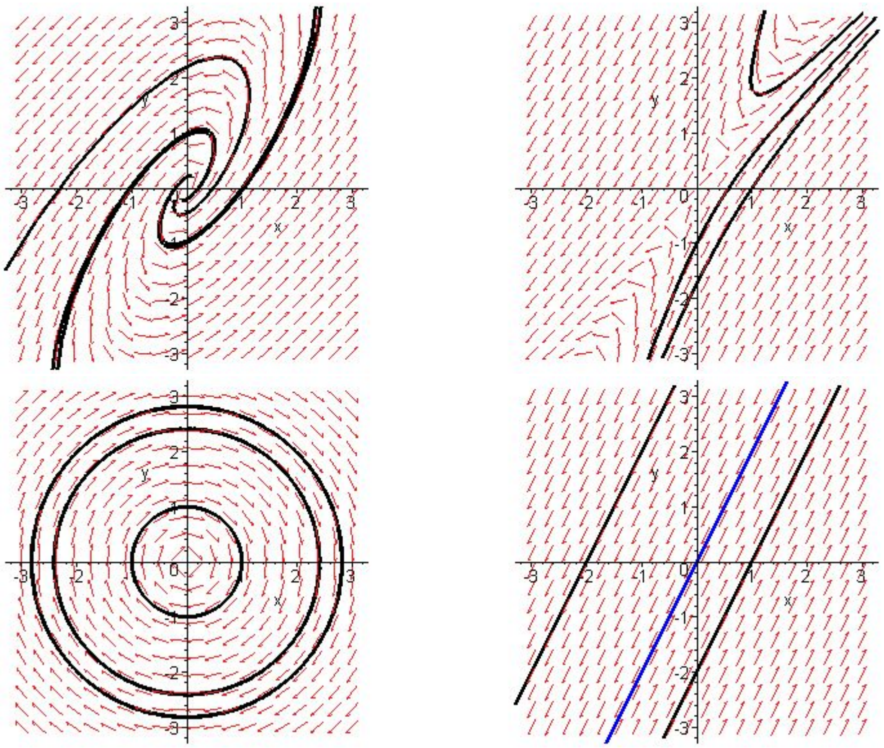

Figure 3: Third set of four phase planes. From top to bottom, Spiral Source, Saddle, Center, "Uniform Motion".

System	Tr(A)	det(A)	Δ	Poincare	λ	V
$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$	2	5	-16	Spiral source	$1+2i$	$\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$
$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$	0	-1	4	Saddle	$-1, 1$	$\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$	0	9	-36	Center	$2i$	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$	0	0	0	Uniform Motion	$0, 0$	$\begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix}$

Homework:

1. Give the solution to the four systems $\mathbf{x}' = A\mathbf{x}$ from the previous page (page 5).

2. For the three systems below $\mathbf{x}' = A\mathbf{x}$, try to give a rough sketch of the behavior of the solutions near the equilibrium (the origin):

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$$

3. For the systems below, use the Poincaré diagram to classify the equilibrium solution. If necessary, first convert the equation to a system of first order equations:

   (a) $\mathbf{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \mathbf{x}$

   (b) $y'' + y' + 3y = 0$

   (c) $2y'' - 3y' + 4y = 0$

   (d) $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$

4. For the following *nonlinear* systems, find the equilibrium solutions (the derivatives are with respect to t, as usual).

   (a) $x' = x - xy$, $y' = y + 2xy$

   (b) $x' = y(2 - x - y)$, $y' = -x - y - 2xy$

5. Explain how the classification of the origin changes by changing the α in the system:

$$\mathbf{x}' = \begin{bmatrix} 0 & \alpha \\ 1 & -2 \end{bmatrix} \mathbf{x}$$