Chapter 3, Computing Solutions

From the theory, we know that every initial value problem:

$$ay'' + by' + cy = g(t) \qquad y(t_0) = y_0 \quad y'(t_0) = v_0$$

has a solution that can be expressed as:

$$y(t) = c_1 y_1 + c_2 y_2 + y_p$$

where y_1, y_2 form a fundamental set of solutions to the homogeneous equation, and $y_p(t)$ is the (particular) solution to the nonhomogeneous equation.

We first consider the homogeneous ODE:

Solving $ay'' + by' + cy = 0$

Form the associated characteristic equation (built by using $y = e^{rt}$ as the ansatz):

$$ar^2 + br + c = 0 \qquad \Rightarrow \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that the solutions depend on the discriminant, $b^2 - 4ac$ in the following way (y_h refers to the solution of the homogeneous equation):

- $b^2 - 4ac > 0 \Rightarrow$ two distinct real roots r_1, r_2. The general solution is:

  $$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

  If $a, b, c > 0$ (as in the Spring-Mass model) we can further say that r_1, r_2 are negative. We would say that this system is OVERDAMPED.

- $b^2 - 4ac = 0 \Rightarrow$ one real root $r = -b/2a$. Then the general solution is:

  $$y_h(t) = e^{-(b/2a)t} \left(C_1 + C_2 t\right)$$

  If $a, b, c > 0$ (as in the Spring-Mass model), the exponential term has a negative exponent. In this case (one real root), the system is CRITICALLY DAMPED.

- $b^2 - 4ac < 0 \Rightarrow$ two complex conjugate solutions, $r = \lambda \pm i\mu$. Then the solution is:

  $$y_h(t) = e^{\lambda t} \left(C_1 \cos(\mu t) + C_2 \sin(\mu t)\right)$$

  If $a, b, c > 0$, then $\lambda < 0$. In the case of complex roots, the system is said to the UNDERDAMPED. If $\lambda = 0$ (this occurs when there is no damping), we get pure periodic motion, with period $2\pi/\mu$.

Solving $y'' + p(t)y' + q(t)y = 0$

Given $y_1(t)$, we can solve for a second linearly independent solution to the homogeneous equation, y_2, by one of two methods:

- By use of the Wronskian: There are two ways to compute this,
  - $W(y_1, y_2) = Ce^{-\int p(t)\,dt}$ (This is from Abel's Theorem)
  - $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

  Therefore, these are equal, and y_2 is the unknown: $y_1 y_2' - y_2 y_1' = Ce^{-\int p(t)\,dt}$

  Summarized Example: $y'' + \frac{2}{t} y' - \frac{2}{t^2} = 0$, with $y_1 = t$.

  By Abel's Theorem: $W(y_1, y_2) = Ce^{-2\ln(t)} = C/t^2$, but we can also compute the Wronskian as: $ty_2' - y_2$. These should be the same: $ty_2' - y_2 = \frac{C}{t^2}$ is a linear first order equation. Solve it and ignore the constant to get that $y_2 = t^{-2}$.

- By Variation of Parameters (the method that the text uses in Section 3.5), where $y_2 = u_2(t) y_1(t)$. See Example 3, p. 171 for an example.

Solving for the particular solution.

Our two methods were: Method of Undetermined Coefficients and Variation of Parameters.

- Method of Undetermined Coefficients

  This method is motivated by the observation that, a linear operator of the form $L(y) = ay'' + by' + cy$, acting on certain classes of functions, returns the same class. In summary, we have Table 3.6.1, reproduced below:

if $g_i(t)$ is:	The ansatz y_{p_i} is:
$P_n(t)$	$t^s(a_0 + a_1 t + \ldots a_n t^n)$
$P_n(t)\mathrm{e}^{\alpha t}$	$t^s \mathrm{e}^{\alpha t}(a_0 + a_1 t + \ldots + a_n t^n)$
$P_n(t)\mathrm{e}^{\alpha t}\sin(\mu t)$ or $\cos(\mu t)$	$t^s \mathrm{e}^{\alpha t}\left((a_0 + a_1 t + \ldots + a_n t^n)\sin(\mu t)\right.$
	$\left. + (b_0 + b_1 t + \ldots + b_n t^n)\cos(\mu t)\right)$

  The t^s term comes from an analysis of the homogeneous part of the solution. That is, multiply by t or t^2 so that no term of the ansatz is included as a term of the homogeneous solution.

- Variation of Parameters: Given $y'' + p(t)y' + q(t)y = g(t)$, with y_1, y_2 solutions to the homogeneous equation, we write the ansatz for the particular solution as:

$$y_p = u_1 y_1 + u_2 y_2$$

  From our analysis, we saw that u_1, u_2 were required to solve:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= 0 \end{aligned}$$

  From which we get the formulas for u_1' and u_2':

$$u_1' = \frac{-y_2 g}{W(y_1, y_2)} \qquad u_2' = \frac{y_1 g}{W(y_1, y_2)}$$

Modeling Oscillations

In the spring-mass system, we saw that the displacement of the spring at time t is governed by the DE:

$$mu'' + \gamma u' + ku = F(t)$$

where m is the mass of the object, γ is the constant corresponding to damping (or friction), and k is the spring constant. Recall that when the object is at rest, its weight and the restorative force of the spring are the same:

$$mg - kL = 0$$

where g is the constant for the acceleration due to gravity (9.8 meters/second^2) and L is the length that the spring was stretched past its natural length.

If $F(t)$ is oscillating and γ is negligible, then we might run into beating or resonance. Beating occurs when the forcing frequency is close to the natural frequency, and resonance occurs when the forcing function is equal to the natural frequency. Recall that the videos showed two tuning forks, we saw the driven spring-mass system and the Tacoma Narrows Bridge.