Chapter 3, Theory

The goal of the theory was to establish the structure of solutions to the second order DE:

$$y'' + p(t)y' + q(t)y = g(t)$$

We saw that two functions form a fundamental set of solutions to the homogeneous DE if they are linearly independent, and we looked at the connection between linear independence and the Wronskian.

1. Vocabulary: Linear Operator, general solution, fundamental set of solutions, linearly independent (and linearly dependent).

2. Theorems:

   - The Existence and Uniqueness Theorem for $y'' + p(t)y' + q(t)y = g(t)$.
   - Principle of Superposition.
   - Linear Independence and the Wronskian:
     - If there is a t_0 such that $W(f, g)(t_0) \neq 0$, then f, g are linearly independent on any interval containing t_0.
     - If $W(f, g) = 0$ for all t, we cannot conclude anything about dependence (for general functions f, g, for example t and $|t|$).
     - If y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$, then the Wronskian is either always zero (y_1, y_2 are linearly dependent), or never zero (y_1, y_2 are linearly independent) on the interval for which the solutions are valid.
       This is Abel's Theorem, which stated that:

       $$W(y_1, y_2)(t) = Ce^{-\int p(t)\,dt}$$

       This quantity is either always zero ($C = 0$) or never zero on the interval for which the solutions are valid.
   - The Fundamental Set of Solutions: $y'' + p(t)y' + q(t)y = 0$
     We can guarantee that we can always find a fundamental set of solutions. We did that by appealing to the Existence and Uniqueness Theorem for the following two initial value problems:
     - y_1 solves $y'' + p(t)y' + q(t)y = 0$ with $y(t_0) = 1, y'(t_0) = 0$
     - y_2 solves $y'' + p(t)y' + q(t)y = 0$ with $y(t_0) = 0, y'(t_0) = 1$

     Note that the choice of initial values simply guarantees that

     $$W(y_1, y_2)(t_0) \neq 0$$

     so we could replace these with other combinations of numbers so that the Wronskian is not zero at t_0.

3. The Structure of Solutions to $y'' + p(t)y' + q(y)y = g(t)$, $y(t_0) = y_0, y'(t_0) = v_0$

   Given a fundamental set of solutions to the homogeneous equation, y_1, y_2, then there is a solution to the initial value problem, written as:

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$$

   where $y_p(t)$ solves the non-homogeneous equation.

   In fact, if we have:

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) + \ldots + g_n(t)$$

   we can solve by splitting the problem up into smaller problems:

   - y_1, y_2 form a fundamental set of solutions to the homogeneous equation.
   - y_{p_1} solves $y'' + p(t)y' + q(t)y = g_1(t)$
   - y_{p_2} solves $y'' + p(t)y' + q(t)y = g_2(t)$
     and so on..
   - y_{p_n} solves $y'' + p(t)y' + q(t)y = g_n(t)$

   and the full solution is:

$$y(t) = C_1 y_1 + C_2 y_2 + y_{p_1} + y_{p_2} + \ldots + y_{p_n}$$

Sections 3.1, 3.4, 3.5 give ways of solving a special homogeneous equation (one with constant coefficients). We also get a method for finding y_2, if we are given y_1 for the more general case.
Sections 3.6 and 3.7 are methods for obtaining the particular part of the solution (Method of Undetermined Coefficients and Variation of Parameters).
Sections 3.8 and 3.9 give a detailed analysis of the solutions using a Spring-Mass system as the physical model.