Solutions to Review Questions: Exam 3

Chapter 6, Sections 5.1-5.3

1. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

   (a)

$$f(t) = \begin{cases} 3, & 0 \le t < 2 \\ 6-t, & t \ge 2 \end{cases}$$

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^2 3e^{-st}\, dt + \int_2^\infty (6-t)e^{-st}\, dt$$

The second antiderivative is found by integration by parts:

$$\int_2^\infty (6-t)e^{-st}\, dt \Rightarrow \quad \begin{array}{ccc} \hline + & 6-t & e^{-st} \\ - & -1 & (-1/s)e^{-st} \\ + & 0 & (1/s^2)e^{-st} \end{array} \quad \Rightarrow \quad e^{-st}\left(-\frac{6-t}{s} + \frac{1}{s^2}\right)\Bigg|_2^\infty$$

Putting it all together,

$$-\frac{3}{s}e^{-st}\Big|_0^2 + \left(0 - e^{-2s}\left(-\frac{4}{s} + \frac{1}{s^2}\right)\right) = -\frac{3e^{-2s}}{s} + \frac{3}{s} + \frac{4e^{-2s}}{s} - \frac{e^{-2s}}{s^2} = \frac{3}{s} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

NOTE: Did you remember to *anti*differentiate in the third column?

   (b)

$$f(t) = \begin{cases} e^{-t}, & 0 \le t < 5 \\ -1, & t \ge 5 \end{cases}$$

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^5 e^{-st}e^{-t}\, dt + \int_5^\infty -e^{-st}\, dt = \int_0^5 e^{-(s+1)t}\, dt + \int_5^\infty -e^{-st}\, dt$$

Taking the antiderivatives,

$$-\frac{1}{s+1}e^{-(s+1)t}\Big|_0^5 + \frac{1}{s}e^{-st}\Big|_5^\infty = \frac{1}{s+1} - \frac{e^{-5(s+1)}}{s+1} + 0 - \frac{e^{-5s}}{s}$$

2. Check your answers to Problem 1 by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.

   (a) $f(t) = 3(u_0(t) - u_2(t)) + (6-t)u_2(t)$

   For the second term, notice that the table entry is for $u_c(t)h(t-c)$. Therefore, if

$$h(t-2) = 6-t \quad \text{then} \quad h(t) = 6-(t+2) = 4-t$$

Therefore, the overall transform is:

$$3\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right) + e^{-2s}\left(\frac{4}{s} - \frac{1}{s^2}\right) = \frac{3}{s} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

(b) $f(t) = \mathrm{e}^{-t}\left(u_0(t) - u_5(t)\right) - u_5(t)$

We can rewrite f in preparation for the transform:

$$f(t) = \mathrm{e}^{-t}u_0(t) - \mathrm{e}^{-t}u_5(t) - u_5(t)$$

For the middle term,

$$h(t-5) = \mathrm{e}^{-t} \quad \Rightarrow \quad h(t) = \mathrm{e}^{-(t+5)} = \mathrm{e}^{-5}\mathrm{e}^{-t}$$

so the overall transform is:

$$F(s) = \frac{1}{s+1} - \mathrm{e}^{-5}\frac{\mathrm{e}^{-5s}}{s+1} - \frac{\mathrm{e}^{-5s}}{s}$$

3. Determine the Laplace transform:

(a) $t^2\mathrm{e}^{-9t}$

$$\frac{2}{(s+9)^3}$$

(b) $\mathrm{e}^{2t} - t^3 - \sin(5t)$

$$\frac{1}{s-2} - \frac{6}{s^4} - \frac{5}{s^2+25}$$

(c) $u_5(t)(t-5)^4$

$$\frac{24\mathrm{e}^{-5s}}{s^5}$$

(d) $\mathrm{e}^{3t}\sin(4t)$

$$\frac{4}{(s-3)^2+16}$$

(e) $\mathrm{e}^t\delta(t-3)$

In this case, notice that $f(t)\delta(t-c)$ is the same as $f(c)\delta(t-c)$, since the delta function is zero everywhere except at $t = c$. Therefore,

$$\mathcal{L}(\mathrm{e}^t\delta(t-c)) = \mathrm{e}^3\mathrm{e}^{-3s}$$

(f) $t^2u_4(t)$

In this case, let $h(t-4) = t^2$, so that

$$h(t) = (t+4)^2 = t^2 + 8t + 16 \quad \Rightarrow \quad H(s) = \frac{2 + 8s + 16s^2}{s^3}$$

and the overall transform is $\mathrm{e}^{-4s}H(s)$.

4. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2-4s+6}$

$$\frac{2s-1}{s^2-4s+6} = \frac{2s-1}{(s^2-4s+4)+2} = 2\frac{s-1/2}{(s-2)^2+2} =$$

$$2\left(\frac{s-2}{(s-2)^2+2} + \frac{3}{2\sqrt{2}}\frac{\sqrt{2}}{(s-2)^2+2}\right) \Rightarrow 2\mathrm{e}^{2t}\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}\mathrm{e}^{2t}\sin(\sqrt{2}t)$$

(b) $\dfrac{7}{(s+3)^3} = \dfrac{7}{2!}\dfrac{2!}{(s+3)^3} \Rightarrow \dfrac{7}{2}t^2\mathrm{e}^{-3t}$

(c) $\dfrac{\mathrm{e}^{-2s}(4s+2)}{(s-1)(s+2)} = \mathrm{e}^{-2s}H(s)$, where

$$H(s) = \frac{4s+2}{(s-1)(s+2)} = \frac{2}{s-1} + \frac{2}{s+2} \qquad \Rightarrow \qquad h(t) = 2\mathrm{e}^{t} + 2\mathrm{e}^{-2t}$$

and the overall inverse: $u_2(t)h(t-2)$.

(d) $\dfrac{3s-2}{2s^2-16s+10}$ Notice that the denominator does not factor "nicely", so we'll go ahead and complete the square with the idea that we'll need hyperbolic sines and cosines in the inverse transform:

$$\frac{3s-2}{2(s^2-8s+5)} = \frac{3}{2}\cdot\frac{s-2/3}{(s-4)^2-11} = \frac{3}{2}\left(\frac{s-4}{(s-4)^2-11} + \frac{10}{3}\cdot\frac{1}{\sqrt{11}}\frac{\sqrt{11}}{(s-4)^2-11}\right)$$

The inverse transform is:

$$\frac{3}{2}\left(\mathrm{e}^{4t}\cosh(\sqrt{11}t) + \frac{10}{3\sqrt{11}}\mathrm{e}^{4t}\sinh(\sqrt{11}t)\right)$$

(e) $\left(\mathrm{e}^{-2s} - \mathrm{e}^{-3s}\right)\dfrac{1}{s^2+s-6} = \left(\mathrm{e}^{-2s} - \mathrm{e}^{-3s}\right)H(s)$

Where:

$$H(s) = \frac{1}{s^2+s-6} = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3}$$

so that

$$h(t) = \frac{1}{5}\mathrm{e}^{2t} - \frac{1}{5}\mathrm{e}^{-3t}$$

and the overall transform is:

$$u_2(t)h(t-2) - u_3(t)h(t-3)$$

NOTE: The denominator was different before; corrected in class.

5. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

(a) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$

$$s^2Y + 1 + 2sY + 2Y = \frac{2}{s^3} + \frac{4}{s^2}$$

so that

$$Y(s) = \frac{2}{s^3(s^2+2s+2)} + \frac{4}{s^2(s^2+2s+2)} - \frac{1}{s^2+2s+2}$$

(b) $y'' + 9y = 10\mathrm{e}^{2t}$, $y(0) = -1$, $y'(0) = 5$

$$s^2Y + s - 5 + 9Y = \frac{10}{s-2} \Rightarrow Y(s) = \frac{10}{(s-2)(s^2+9)} - \frac{s-5}{s^2+9}$$

(c) $y'' - 4y' + 4y = t^2 e^t$, $y(0) = 0$, $y'(0) = 0$

$$(s^2 - 4s + 4)Y = \frac{2}{(s-1)^3} \Rightarrow Y(s) = \frac{2}{(s-1)^3(s-2)^2}$$

6. Solve the given initial value problems using Laplace transforms:

(a) $2y'' + y' + 2y = \delta(t-5)$, zero initial conditions.

$$Y = \frac{e^{-5s}}{2s^2 + s + 2} = e^{-5s} H(s)$$

where

$$H(s) = \frac{1}{2s^2 + s + 2} = \frac{1}{2}\frac{1}{s^2 + \frac{1}{2}s + 1} = \frac{1}{2}\frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}} = \frac{1}{2}\frac{4}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}$$

Therefore,

$$h(t) = \frac{2}{\sqrt{15}} e^{-1/4\,t} \sin\left(\frac{\sqrt{15}}{4} t\right)$$

And the overall solution is $u_5(t)h(t-5)$

(b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$

$$s^2 Y + 3s - 10 + 6(sY + 3) + 9Y = 0 \quad \Rightarrow \quad Y = -\frac{3s+8}{(s+3)^2}$$

Partial Fractions:

$$-\frac{3s+8}{(s+3)^2} = -\frac{3}{(s+3)} + \frac{1}{(s+3)^2} \Rightarrow y(t) = -3e^{-3t} + te^{-3t}$$

(c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$

$$Y = e^{-s}\frac{1}{s(s-3)(s+1)} - \frac{1}{(s+1)(s-3)} = e^{-s}H(s) - \frac{1}{4}\frac{1}{s-3} + \frac{1}{4}\frac{1}{s+1}$$

where

$$H(s) = \frac{1}{s(s-3)(s+1)} = -\frac{1}{3}\frac{1}{s} + \frac{1}{12}\frac{1}{s-3} + \frac{1}{4}\frac{1}{s+1}$$

so that

$$h(t) = -\frac{1}{3} + \frac{1}{12}e^{3t} + \frac{1}{4}e^{-t}$$

and the overall solution is:

$$y(t) = -\frac{1}{4}e^{3t} + \frac{1}{4}e^{-t} + u_1(t)h(t-1)$$

(d) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$

$$Y = e^{-\pi/2\,s}\frac{1}{s^2+4} + \frac{1}{s^2+4}$$

Therefore,

$$y(t) = \frac{1}{2}\sin(2t) + u_{\pi/2}(t)\frac{1}{2}\sin(2(t - \pi/2))$$

(e) $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi)$, $y(0) = y'(0) = 0$. Write your answer in piecewise form.

$$Y(s) = \sum_{k=1}^{\infty} \mathrm{e}^{-2k\pi s} \frac{1}{s^2+1}$$

Therefore, term-by-term,

$$y(t) = \sum_{k=1}^{\infty} u_{2k\pi}(t)\sin(t - 2\pi k) = \sum_{k=1}^{\infty} u_{2\pi k}(t)\sin(t)$$

Piecewise,

$$y(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 2\pi \\ \sin(t) & \text{if} \quad 2\pi \le t < 4\pi \\ 2\sin(t) & \text{if} \quad 4\pi \le t < 6\pi \\ 3\sin(t) & \text{if} \quad 6\pi \le t < 8\pi \\ \vdots & \vdots \end{cases}$$

7. Short Answer:

(a) $\int_0^{\infty} \sin(3t)\delta(t - \frac{\pi}{2})\, dt = \sin(3\pi/2) = -1$, since

$$\int_0^{\infty} f(t)\delta(t-c)\, dt = f(c)$$

(b) If $y'' + 2y' + 3y = 0$ and $y(0) = 1$, $y'(0) = -1$, compute $y''(0)$, $y'''(0)$, and $y^{(4)}(0)$.
We see that:

$$y'' = -2y' - 3y \text{ at } x = 0 \Rightarrow y''(0) = -2(-1) - 3(1) = -1$$

$$y''' = -2y'' - 3y' \text{ at } x = 0 \Rightarrow y'''(0) = (-2)(-1) - 3(-1) = 5$$
$$y^{(4)} = -2y''' - 3y'' \text{ at } x = 0 \Rightarrow y^{(4)}(0) = (-2)(5) - 3(-1) = -7$$

(c) Using your previous result, give the Taylor expansion of the solution to the differential equation using at least 5 terms.

$$y(x) = 1 - x - \frac{1}{2!}x^2 + \frac{5}{3!}x^3 + \frac{7}{4!}x^4 + \ldots$$

(d) If $y'(t) = \delta(t - c)$, what is $y(t)$?
We could solve formally using Laplace transforms:

$$sY - y(0) = \mathrm{e}^{-cs} \Rightarrow Y = \frac{\mathrm{e}^{-cs}}{s} + \frac{y(0)}{s}$$

so that $y(t) = u_c(t) + y(0)$, where $y(0)$ we can take to be an arbitrary constant.

(e) What is the expected radius of convergence for the series expansion of $f(x) = 1/(x^2+2x+5)$ if the series is based at $x_0 = 1$?

The roots of the denominator are where $x^2+2x+5=0$. Use the quadratic formula or complete the square to find the roots,

$$(x+1)^2 = -4 \Rightarrow x = -1 \pm 2i$$

Find the distance (in the complex plane) between $x = 1$ and either root (the distances will be the same). In this case,

$$\rho = \sqrt{2^2+2^2} = \sqrt{8}$$

(f) Use Laplace transforms to solve for $F(s)$, if

$$f(t) + 2\int_0^t \cos(t-x) f(x)\, dx = \mathrm{e}^{-t}$$

(So only solve for the transform of $f(t)$, don't invert it back).

$$F(s) + 2F(s)\frac{s}{s^2+1} = \frac{1}{s+1} \quad \Rightarrow \quad F(s)\left(\frac{(s+1)^2}{s^2+1}\right) = \frac{1}{s+1}$$

so that

$$F(s) = \frac{s^2+1}{(s+1)^3}$$

(g) In order for the Laplace transform of f to exist, f must be?

f must be piecewise continuous and of exponential order

(h) Can we assume that the solution to: $y'' + p(x)y' + q(x)y = u_3(x)$ is a power series?

No. Notice that the second derivative is not continuous at $x = 3$, but the second derivative of the power series would be.

8. More on Laplace Transforms:

(a) Your friend tells you that the solutions to the IVPs:

$$y''+2y'+y = 0,\ \ y(0) = 0,\ y'(0) = 1 \qquad \text{and} \qquad y''+2y'+y = \delta(t)\ \ y(0) = 0,\ y'(0) = 0$$

are exactly the same. Are they really? Explain.

Both models give the same solution if $t \geq 0$. If we consider all time, then the solutions are different.

Conceptually, the two IVPs are also modeling different behavior. In the second IVP, we are modeling a "hit" at time zero, but in the first, the spring-mass system (for example), is simply going through equilibrium at a velocity of 1.

By the way, the solution to both IVPs is

$$y(t) = t\mathrm{e}^{-t}$$

Valid for all positive time in both models, valid for all time in the first, only valid for $t \geq 0$ in the second (in the Dirac model, the function would be zero for all negative time due to the initial conditions).

(b) Let $f(t) = t$ and $g(t) = u_2(t)$.

i. Use the Convolution Theorem to compute $f * g$.
To use the Convolution Theorem,

$$\mathcal{L}(t * u_2(t)) = \frac{1}{s^2} \cdot \frac{e^{-2s}}{s} = e^{-2s}\frac{1}{s^3} = e^{-2s}H(s)$$

so that $h(t) = \frac{1}{2}t^2$. The inverse transform is then

$$u_2(t)\frac{1}{2}(t-2)^2$$

ii. Verify your answer by directly computing the integral.
By direct computation, we'll choose to "flip and shift" the function t:

$$f * g = \int_0^t (t-x)u_2(x)\,dx$$

Notice that $u_2(x)$ is zero until $x = 2$, then $u_2(x) = 1$. Therefore, if $t \leq 2$, the integral is zero. If $t \geq 2$, then:

$$\int_0^t (t-x)u_2(x)\,dx = \int_2^t t - x\,dx = tx - \frac{1}{2}x^2\Big|_2^t = t^2 - \frac{1}{2}t^2 - 2t + 2 = \frac{1}{2}(t-2)^2$$

valid for $t \geq 2$, zero before that. This means that the convolution is:

$$t * u_2(t) = \frac{1}{2}(t-2)^2 u_2(t)$$

9. Find the recurrence relation between the coefficients for the power series solutions to the following:

(a) $2y'' + xy' + 3y = 0$, $x_0 = 0$.
Substituting our power series in for y, y', y'':

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

Noting that in the second sum we could start at $n = 0$, since the first term (constant term) would be zero anyway, we can start all series with a constant term:

$$\sum_{k=0}^{\infty} \left(2(k+2)(k+1)a_{k+2} + ka_k + 3a_k\right) x^k = 0$$

From which we get the recurrence relation:

$$a_{k+2} = -\frac{k+3}{2(k+2)(k+1)}\,a_k$$

(b) $(1-x)y'' + xy' - y = 0$, $x_0 = 0$

Substituting our power series in for y, y', y'':

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

The two middle sums can have their respective index taken down by one (so that formally the series would start with $0x^0$):

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Now make all the indices the same. To do this, in the first sum make $k = n-2$, in the second sum take $k = n-1$. Doing this and collecting terms:

$$\sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - (k+1)k a_{k+1} + (k-1)a_k\right) x^k = 0$$

So we get the recursion:

$$a_{k+2} = \frac{(k+1)k\, a_{k+1} - (k-1)a_k}{(k+2)(k+1)}$$

(c) $y'' - xy' - y = 0$, $x_0 = 1$

Done in class;

$$a_{n+2} = \frac{1}{n+2}\left(a_{n+1} + a_n\right)$$

10. Find the first 5 terms of the power series solution to $\mathrm{e}^x y'' + xy = 0$ if $y(0) = 1$ and $y'(0) = -1$.

Compute the derivatives directly, then (don't forget to divide by $n!$):

$$y(x) = 1 - x - \frac{1}{6}x^3 + \frac{1}{6}x^4 + \ldots$$

11. Find the radius of convergence for the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}x^n \quad \rho = 1$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}(x+3)^n \quad \rho = 1/2$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n} \quad \rho = \mathrm{e}$ (From the HW)