Solutions: Section 2.6

1. Problem 1: $(2x+3)+(2y-2)y' = 0$

   We want $f_x = M(x,y) = 2x+3$ and $f_y = N(x,y) = 2y-2$. We check if this is possible:

   $$M_y = 0 \qquad N_x = 0$$

   Now antidifferentiate M with respect to x:

   $$f(x,y) = \int M(x,y)\,dx = \int 2x+3\,dx = x^2+3x+g(y)$$

   where g is some unknown function of y. Two ways of proceeding (which are equivalent). I'll list both methods for this problem:

   - Try to get f from N, and compare:

     $$f(x,y) = \int N(x,y)\,dy = \int 2y-2\,dy = y^2-2y+\hat{g}(x)$$

     where $\hat{g}$ is an unknown function of x. Comparing this to what we had before, we see that:

     $$f(x,y) = x^2+3x+y^2-2y$$

     so the implicit solution is: $x^2+3x+y^2-2y = C$

     NOTE: You can always check your answer!
   - Another method: Starting from where we left off,

     $$f(x,y) = x^2+3x+g(y)$$

     we can see what g needs to be in order for $f_y = N$, or:

     $$f_y = g'(y) = 2y-2 = N$$

     In that case, $g(y) = y^2-2y$, and $f(x,y) = x^2+3x+y^2-2y$.

     The implicit solution is: $x^2+3x+y^2-2y = C$

2. Problem 3: $(3x^2-2xy+2)\,dx+(6y^2-x^2+3)\,dy = 0$

   Check to see if the equation is exact:

   $$M_y = -2x \qquad N_x = -2x$$

   So yes. Now we'll antidifferentiate M with respect to x:

   $$f(x,y) = \int M\,dx = \int 3x^2-2xy+2\,dx = x^3-x^2y+2x+g(y)$$

   Check to see if f_y is equal to N:

   $$f_y = -x^2+g'(y) = 6y^2-x^2+3$$

   so that $g'(y) = 6y^2+3$. That gives $g(y) = 2y^3+3y$. Put this back in to get the full solution, $f(x,y) = c$:

   $$x^3-x^2y+2x+2y^3+3y = C$$

3. Problem 4: $(2xy^2 + 2y) + (2x^2y + 2x)\frac{dy}{dx} = 0$

   Check for "exactness":

$$M_y = 4xy + 2 \qquad N_x = 4xy + 2$$

   Now set:

$$f(x, y) = \int M\,dx = \int 2xy^2 + 2y\,dx = x^2y^2 + 2xy + g(y)$$

   And check to see that $f_y = N$:

$$f_y = 2x^2y + 2x + g'(y) = 2x^2y + 2x$$

   In this case, $g'(y) = 0$, and we don't need to add $g(y)$. The implicit solution:

$$x^2y^2 + 2xy = C$$

4. Problem 13: $(2x - y)\,dx + (2y - x)\,dy = 0$

   Check first: $M_y = -1 = N_x$, so the DE is exact.

   Now,

$$f(x, y) = \int M\,dx = \int 2x - y\,dx = x^2 - xy + g(y)$$

   where we check f_y to make it equal to $N(x, y)$:

$$f_y = -x + g'(y) = 2y - x \quad \Rightarrow \quad g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2$$

   Our implicit solution is:

$$x^2 - xy + y^2 = C$$

   With the initial condition $y(1) = 3$, we get:

$$1^2 - (1)(3) + 3^2 = C \quad \Rightarrow \quad C = 7$$

   The solution to the IVP is:

$$x^2 - xy + y^2 = 7$$

   There are a couple of ways we can determine the interval on which the solution is valid (recall that this is a rotated ellipse). Here are two methods:

   - We notice that:

$$y' = \frac{y - 2x}{2y - x}$$

     so the solution $y(x)$ will have a vertical tangent where $2y = x$ or $y = \frac{1}{2}x$. Looking for where this occurs on our solution:

$$x^2 - xy + y^2 = 7$$

we get:

$$x^2 - x\left(\frac{1}{2}x\right) + \left(\frac{1}{2}x\right)^2 = 7 \quad \Rightarrow \quad x = \pm\sqrt{283}$$

We will take the inside interval since the initial $x_0 = 1$

$$x \in \left(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}\right)$$

- Method 2:
  We can isolate y and find the restrictions on x (think quadratic formula in y, parentheses added for emphasis):

  $$y^2 + (-x)\, y + \left(x^2 - 7\right) = 0$$

  Now,

  $$y = \frac{x \pm \sqrt{x^2 - 4(x^2 - 7)}}{2} = \frac{x \pm \sqrt{28 - 3x^2}}{2}$$

  Take the positive root since $y(1) = 3$.
  The restriction on x would be that $28 - 3x^2 \geq 0$. Therefore,

  $$-\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}}$$

- Problem 15: $(xy^2 + bx^2y)\, dx + (x + y)x^2\, dy = 0$
  First, for this to be exact:

  $$M_y = 2xy + bx^2 = 3x^2 + 2xy = N_x$$

  So $b = 3$. With this, find the solution to the DE:

  $$f(x, y) = \int M\, dx = \int xy^2 + 3x^2y\, dx = \frac{1}{2}x^2y^2 + x^3y + g(y)$$

  And solve for $g(y)$:

  $$f_y = x^2y + x^3 + g'(y) = x^3 + x^2y$$

  So we didn't need $g(y)$. This leaves:

  $$\frac{1}{2}x^2y^2 + x^3y = C$$

5. Problem 18: Done in Class. The idea is:

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Given that, $M_y = 0$ and $N_x = 0$, so the equation is exact.

We would solve:

$$f(x, y) = \int M(x)\,dx + g(y)$$

Taking f_y and equating it to N: $f_y = g'(y) = N(y)$. Therefore, $g(y) = \int N(y)\,dy$, and the solution is:

$$f(x, y) = \int M(x)\,dx + \int N(y)\,dy = C$$

which is what we do with separable equations.

6. Problem 19: We see that:

$$x^2y^3 + (x + xy^2)\frac{dy}{dx} = 0$$

is not separable, since $(x^2y^3)_y = 3x^2y^2$, but $(x + xy^2)_x = 1 + y^2$. However, after multiplication by $\mu(x, y) = 1/(xy^3)$, we get:

$$\frac{x^2y^3}{xy^3} + \frac{x(1 + y^2)}{xy^3}\frac{dy}{dx} = 0$$

Simplify:

$$x + (y^{-3} + y^{-1})\frac{dy}{dx} = 0$$

Note that this becomes *separable*, so it is also exact.

Using the methods from this section,

$$f(x, y) = \int M\,dx = \frac{1}{2}x^2 + g(y)$$

and $f_y = g'(y) = N = y^{-3} + y^{-1}$. Therefore,

$$g(y) = \int y^{-3} + \frac{1}{y}\,dy = -\frac{1}{2}y^{-2} + \ln(y)$$

The solution is:

$$\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln(y) = C$$

7. Problem 22:

In this case, when we multiply by the given integrating factor:

$$xe^x(x + 2)\sin(y) + xe^x x\cos(y) = 0$$

Expand it to make the partial derivatives a bit easier:

$$x^2e^x\sin(y) + 2xe^x\sin(y) + (x^2e^x\cos(y))\frac{dy}{dx} = 0$$

Now check the partials:

$$M_y = x^2 e^x \cos(y) + 2xe^x \cos(y)$$

and

$$N_x = \cos(y)\left(2xe^x + x^2 e^x\right)$$

And these are the same (so it is exact). In fact, this equation is also separable (this is done for fun):

$$\frac{dy}{dx} = -\frac{\sin(y)(x^2+2x)e^x}{\cos(y)x^2 e^x} = -\tan(y)\left(1+\frac{2}{x}\right) \quad \Rightarrow \quad \int \cot(y)\,dy = -\int 1 + \frac{2}{x}\,dx$$

Let's see if we can avoid[1] the antiderivative of $\cot(y)$:

$$f(x,y) = \int M\,dx = \sin(y)\int e^x(x^2+2x)\,dx + g(y) = x^2 e^x \sin(y) + g(y)$$

Now check $f_y = N$:

$$f_y = x^2 e^x \cos(y) + g'(y) = x^2 e^x \cos(y)$$

so we did not need $g(y)$ in this case. The solution is:

$$x^2 e^x \sin(y) = C$$

[1]Actually, this is not bad- Do it with a u, du substitution.