Selected Problems from Section 3.6

For problems 19-22, be sure you can write the form of the particular solution. You do not need to solve for any of the coefficients.

1. Problem 19: Find the homogeneous solution first.

$$r = 0, 3 \quad \Rightarrow \quad y_h(t) = C_1 + C_2 e^{3t}$$

   We will make three particular guesses, one for each class of forcing function shown:

   - $g_1(t) = 2t^4$, so y_{p_1} will be a fourth degree polynomial. Multiply it by t because constant solutions are part of the homogeneous solution (C_1).

$$y_{p_1} = t\left(A_1 + A_2 t + A_3 t^2 + A_4 t^3 + A_5 t^4\right)$$

   - $g_2(t) = t^2 e^{-3t}$, so y_{p_2} will be a quadratic times the exponential. Multiply it by t because e^{-3t} is part of the homogeneous solution:

$$y_{p_2} = t\left(B_1 + B_2 t + B_3 t^2\right) e^{-3t}$$

   - $g_3(t) = \sin(3t)$, so y_{p_2} will be:

$$y_{p_2} = C_1 \cos(3t) + C_2 \sin(3t)$$

2. Problem 20: Find the homogeneous solution first.

$$r = \pm i \quad \Rightarrow \quad y_h(t) = C_1 \cos(t) + C_2 \sin(t)$$

   We will make two particular guesses, one for each class of forcing function shown (Rewrite as $t + t\sin(t)$):

   - $g_1(t) = t$, so $y_{p_1} = At + B$
   - $g_2(t) = t\sin(t)$, so y_{p_2} will be a linear function times a sine and cosine. We will have to multiply by t since $\sin(t)$ and $\cos(t)$ are parts of the homogenous solution:

$$y_{p_2} = t\left[(A_1 t + A_2)\cos(t) + (A_3 t + A_4)\sin(t)\right]$$

3. Problem 28: Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin(m\pi t)$$

   where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, 2, \ldots, N$. (Note: This type of problem does arise from some engineering models!)

Writing this out, we see:

$$y'' + \lambda^2 y = a_1 \sin(\pi t) + a_2 \sin(2\pi t) + a_2 \sin(3\pi t) + \ldots + a_N \sin(N\pi t)$$

We solve this piece by piece; first get the homogeneous part of the solution:

$$y_h(t) = C_1 \cos(\lambda t) + C_2 \sin(\lambda t)$$

Now the m^{th} function $g(t)$ is given by: $a_m \sin(m\pi t)$, so our m^{th} guess is:

$$y_p = A \cos(m\pi t) + B \sin(m\pi t)$$

Substitute this into the differential equation, where

$$y_p'' = -Am^2\pi^2 \cos(m\pi t) - Bm^2\pi^2 \sin(m\pi t)$$

so $y_p'' + \lambda^2 y_p$ is:

$$A(\lambda^2 - m^2\pi^2) \cos(m\pi t) + B(\lambda^2 - m^2\pi^2) \sin(m\pi t)$$

which is where the text answer comes from.

4. Problem 31 continues from Problem 38 in Section 3.5.

   If Y_1 and Y_2 are solutions to:

$$ay'' + by' + cy = g(t)$$

   with $a, b, c > 0$, and if y_1, y_2 form a fundamental set to the homogeneous equation, then:

   - In problem 38, Sect 3.5, we said that $c_1 y_1 + c_2 y_2 \to 0$ as $t \to \infty$.
   - In this section, we said that $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$.

   By the previous two items, $Y_1 - Y_2 \to 0$ as $t \to \infty$

   If $b = 0$, then $c_1 y_1 + c_2 y_2$ do not go to zero, but oscillate. Therefore, $Y_1 - Y_2$ would oscillate (but remain bounded) as $t \to \infty$.

5. Problem 32: If $g(t) = d$, write the solution:

$$ay'' + by' + cy = d$$

   The homogeneous solution is $C_1 y_1 + C_2 y_2$ (neither y_1 nor y_2 are constant if $a, b, c > 0$).

   The ansatz for the particular part of the solution would be

$$y_p = A$$

Which, when substituted into the differential equation, gives $A = d/c$. The full solution is therefore

$$y(t) = c_1 y_1 + c_2 y_2 + \frac{d}{c}$$

By problem 38 of Section 3.5, $y(t) \to d/c$ as $t \to \infty$.

If $c = 0$, the solutions to the characteristic equation are

$$r = 0, -b/a$$

so the homogeneous part of the solution is $C_1 + C_2 \mathrm{e}^{-(b/a)t}$. In this case, the ansatz for the particular part of the solution needs to be multiplied by t,

$$y_p = At$$

Substitution into the DE gives us that $y_p = \dfrac{d}{b} t$

Finally, if $b = 0$ as well, the differential equation just becomes

$$ay'' = d \quad \Rightarrow \quad y'' = \frac{d}{a} \quad \Rightarrow \quad y' = \frac{d}{a} t + C_1 \quad \Rightarrow \quad y = \frac{d}{2a} t^2 + C_1 t + C_2$$

We see that the homogeneous part of the solution is $C_1 t + C_2$ and the particular part of the solution becomes

$$y_p(t) = \frac{d}{2a} t^2$$