Selected Problems from Section 3.7

1. Problem 3: $y'' + 2y' + y = 3\mathrm{e}^{-t}$

   Using Variation of Parameters:

   $$y_1 = \mathrm{e}^{-t} \quad y_2 = t\mathrm{e}^{-t} \quad g(t) = 3\mathrm{e}^{-t} \quad W = \mathrm{e}^{-2t}$$

   so

   $$u_1' = \frac{-3t\mathrm{e}^{-t}\mathrm{e}^{-t}}{\mathrm{e}^{-2t}} = -3t \quad \Rightarrow \quad u_1(t) = -\frac{3}{2}t^2$$

   $$u_2' = \frac{\mathrm{e}^{-t}3\mathrm{e}^{-t}}{\mathrm{e}^{-2t}} = 3 \quad \Rightarrow \quad u_2(t) = 3t$$

   Therefore, the particular solution is $y_p = u_1y_1 + u_2y_2$:

   $$y_p(t) = -\frac{3}{2}t^2\mathrm{e}^{-t} + 3t^2\mathrm{e}^{-t} = \frac{3}{2}t^2\mathrm{e}^{-t}$$

   Using the method of undetermined coefficients, we would have initially guessed that y_p was a constant times the exponential function, but we have to multiply by t^2. Do that, and differentiate to substitute back into the ODE:

   $$y_p = At^2\mathrm{e}^{-t} \quad y_p' = A(2t - t^2)\mathrm{e}^{-t} \quad y_p'' = A(2 - 4t + t^2)\mathrm{e}^{-t}$$

   We end up with $A = \frac{3}{2}$, which is what we had using Variation of Parameters.

2. Problem 5: $y'' + y = \tan(t)$

   $$y_1 = \cos(t) \quad y_2 = \sin(t) \quad g(t) = \tan(t) \quad W = 1$$

   Therefore, we get an integral that can be a bit tricky, but we can simplify the integrand first:

   $$u_1' = -\sin(t)\tan(t) = -\frac{\sin^2(t)}{\cos(t)} = \frac{-1 + \cos^2(t)}{\cos(t)} = -\sec(t) + \cos(t)$$

   so that $u_1 = -\ln|\sec(t) + \tan(t)| + \sin(t)$.

   $$u_2' = \cos(t)\tan(t) \quad \Rightarrow \quad u_2 = \int \sin(t)\,dt = -\cos(t)$$

   Therefore,

   $$y_p = (-\ln|\sec(t) + \tan(t)| + \sin(t))\cos(t) - \cos(t)\sin(t) =$$

   $$-\cos(t)\ln|\sec(t) + \tan(t)|$$

3. Problem 15: We won't verify that y_1, y_2 are solutions, but you should. Assuming that they are (remember to put the equation in standard form by dividing by t):

$$y_1 = 1 + t \quad y_2 = \mathrm{e}^t \quad g(t) = t\mathrm{e}^{2t} \quad W = t\mathrm{e}^t$$

$$u_1' = \frac{-\mathrm{e}^t t\mathrm{e}^{2t}}{t\mathrm{e}^t} = -\mathrm{e}^{2t} \quad \Rightarrow \quad u_1(t) = -\frac{1}{2}\mathrm{e}^{2t}$$

For u_2, we'll integrate by parts:

$$u_2' = \frac{(1+t)t\mathrm{e}^{2t}}{t\mathrm{e}^t} = (1+t)\mathrm{e}^t \quad \Rightarrow \quad u_2(t) = t\mathrm{e}^t$$

The particular solution is $u_1 y_1 + u_2 y_2$ which simplifies to

$$y_p = \frac{1}{2}\mathrm{e}^{2t}\,(t-1)$$

4. Problem 17 comes out nicely as well- Be sure to get standard form before going too far. Use $u = \ln(x)$ for the integrals.