Selected Solutions, Section 6.3

1. Problem 2: To sketch the graph, try first rewriting what is given as a piecewise defined function. The function:
$$(t-3)u_2(t) - (t-2)u_3(t)$$
depends on the value of t:

   - If $t < 2$, the function is zero, since u_2 and u_3 are zero.
   - If $2 \le t < 3$, then the function is just $t-3$, since $u_3(t)$ is still zero.
   - If $t \ge 3$, then the function would be $t - 3 - (t-2) = -3 + 2 = -1$, since both u_2 and u_3 would simplify to 1.

   Now it's easy to graph.

2. Problem 3: Plot $(t-\pi)^2 u_\pi(t)$. This is the right half of t^2 shifted π units to the right.

3. Problem 6: Break it up:
$$(t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t) = \begin{cases} 0 & \text{if } t < 1 \\ (t-1) & \text{if } 1 \le t < 2 \\ (t-1) - 2(t-2) & \text{if } 2 \le t < 3 \\ (t-1) - 2(t-2) + (t-3) & \text{if } t \ge 3 \end{cases}$$
Rewriting this, we get:
$$= \begin{cases} 0 & \text{if } t < 0 \\ t-1 & \text{if } 1 \le t < 2 \\ -t+3 & \text{if } 2 \le t < 3 \\ 0 & \text{if } t \ge 3 \end{cases}$$

4. Problem 8: To use the table, we need that:
$$u_1(t)f(t-1) = u_1(t)(t^2 - 2t + 2)$$
so that $f(t-1) = t^2 - 2t + 2$. This means that
$$f(t) = (t+1)^2 - 2(t+1) + 2 = t^2 + 2t + 1 - 2t - 2 + 2 = t^2 + 1$$
Now the table entry says:

$f(t)$	$F(s)$
$u_c(t)f(t-c)$	$\mathrm{e}^{-sc}F(s)$

   so we see that
$$F(s) = \mathcal{L}(t^2 + 1) = \frac{2}{s^3} + \frac{1}{s}$$
so that our final answer is: $\mathrm{e}^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right)$
We could verify it directly as well by computing
$$\mathcal{L}(u_1(t)(t^2 - 2t + 2)) = \int_1^\infty \mathrm{e}^{-st}(t^2 - 2t + 2)\, dt$$

5. Problem 11: Find the Laplace transform of

$$f(t) = (t-3)u_2(t) - (t-2)u_3(t)$$

We break it up using the linearity of $\mathcal{L}$:

- To use the table, we must have: $(t-3)u_2(t) = f(t-2)u_2(t)$. Therefore,

$$f(t) = t + 2 - 3 = t - 1 \qquad F(s) = \frac{1}{s^2} - \frac{1}{s}$$

  and the Laplace transform of $f(t-2)u_2(t)$ is $\mathrm{e}^{-sc}F(s)$, so overall, we get:

$$\mathrm{e}^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right)$$

- Similarly, for the second term, $(t-2)u_3(t) = f(t-3)u_3(t)$, so

$$f(t) = t + 3 - 2 = t + 1 \qquad F(s) = \frac{1}{s^2} + \frac{1}{s}$$

  and overall, the Laplace transform of this part is:

$$\mathrm{e}^{-3s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

Overall, we subtract the two answers:

$$F(s) = \mathrm{e}^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right) - \mathrm{e}^{-3s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

6. Problem 14: Find the inverse Laplace transform of

$$F(s) = \mathrm{e}^{-2s} \cdot \frac{1}{s^2 + s - 2}$$

This is of the form $\mathrm{e}^{-sc}F(s)$, so we need to find the inverse transform of $1/(s^2+s-2)$. We'll do this by Partial Fraction Decomposition. This F refers to the table, not the original F:

$$F(s) = \frac{1}{s^2+s-2} = \frac{A}{s+2} + \frac{B}{s-1} = -\frac{1}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}$$

and

$$f(t) = -\frac{1}{3}\mathrm{e}^{-2t} + \frac{1}{3}\mathrm{e}^{t}$$

So our overall answer is $u_2(t)f(t-2)$, or:

$$u_2(t)\left(-\frac{1}{3}\mathrm{e}^{-2(t-2)} + \frac{1}{3}\mathrm{e}^{t-2}\right)$$

7. Problem 15: Find the inverse Laplace transform of

$$G(s) = \frac{2e^{-2s}(s-1)}{s^2 - 2s + 2}$$

(I changed the notation of the original function so as not to confuse $F(s)$ in the table with $F(s)$ in the original question).

We will rewrite this expression, keeping the table in mind:

$$G(s) = 2e^{-2s}\frac{s-1}{s^2-2s+2} = 2e^{-2s}\frac{s-1}{(s-1)^2+1} = 2e^{-sc}F(s)$$

We see that, given this $F(s)$, then $f(t) = e^t \cos(t)$ and our overall inverse Laplace transform is:

$$2u_2(t)f(t-2) = 2u_2(t)e^{t-2}\cos(t-2)$$

8. Problem 18: Rewrite as:

$$G(s) = e^{-s}\frac{1}{s} + e^{-2s}\frac{1}{s} - e^{-3s}\frac{1}{s} - e^{-4s}\frac{1}{s}$$

Each of these is in the form $e^{-sc}F(s)$, where $F(s) = 1/s$, so $f(t) = 1$. Notice that $f(t-k) = 1$ as well, so that the overall inverse is simply:

$$g(t) = u_1(t) + u_2(t) - u_3(t) - u_4(t)$$

9. Problem 27: The Laplace transform is a linear operator, so:

$$\mathcal{L}\left(1 + \sum_{k=1}^{\infty}(-1)^k u_k(t)\right) = \mathcal{L}(1) + \sum_{k=1}^{\infty}(-1)^k \mathcal{L}(u_k(t)) = \frac{1}{s} + \sum_{k=1}^{\infty}(-1)^k\frac{e^{-ks}}{s}$$

This reminds us of a geometric series. Let's put it in that form:

$$= \frac{1}{s}\left(1 + \sum_{k=1}^{\infty}(-e^{-s})^k\right) = \frac{1}{s}\cdot\frac{1}{1+e^{-s}}$$

10. Problem 28: The answer we want might again remind us of a geometric series. If f is periodic with period T, then:

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st}f(t)\,dt = \int_0^T e^{-st}f(t)\,dt + \int_T^{2T} e^{-st}f(t)\,dt + \ldots + \int_{kT}^{(k+1)T} e^{-st}f(t)\,dt + \ldots$$

For each of the terms in the sum, perform a u, du substitution so that $u = t - kT$, $du = dt$:

$$\int_{kT}^{(k+1)T} e^{-st}f(t)\,dt = \int_0^T e^{-s(u+kT)}f(u+kT)\,du = e^{-kT}\int_0^T e^{-su}f(u)\,du$$

This factors out of the sum:

$$\int_0^T e^{-su}f(u)\,du\left(1 + e^{-sT} + e^{-2sT} + \ldots + e^{-ksT} + \ldots\right) = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$

11. Problem 29: Use the results of Problem 28- If f is periodic with period T, then

$$\mathcal{L}(f(t)) = \frac{\int_0^T \mathrm{e}^{-st} f(t)\, dt}{1 - \mathrm{e}^{-sT}}$$

In this case, $T = 2$, and

$$\int_0^2 \mathrm{e}^{-st} f(t)\, dt = \int_0^1 \mathrm{e}^{-st}\, dt = -\frac{1}{s}\mathrm{e}^{-st}\Big|_0^1 = -\frac{\mathrm{e}^{-s}}{s} + \frac{1}{s} = \frac{1}{s}\left(1 - \mathrm{e}^{-s}\right)$$

Put this into the formula:

$$\mathcal{L}(f(t)) = \frac{\frac{1}{s}(1 - \mathrm{e}^{-s})}{1 - \mathrm{e}^{-2s}}$$

This doesn't seem to be the same answer as in Problem 27; However we might note that:

$$1 - \mathrm{e}^{-2s} = 1^2 - \left(\mathrm{e}^{-s}\right)^2 = \left(1 - \mathrm{e}^{-s}\right)\left(1 + \mathrm{e}^{-s}\right)$$