Selected Solutions, Section 6.5

For the graphs of 1-8, see the Maple Worksheet on our class website.

1. Problem 1: $y'' + 2y' + 2y = \delta(t - \pi), y(0) = 1, y'(0) = 0$

$$s^2Y - s + 2(sY - 1) + 2Y = \mathrm{e}^{-\pi s}$$

$$(s^2 + 2s + 2)Y = \mathrm{e}^{-\pi s} + s + 2 \quad \Rightarrow \quad Y = \mathrm{e}^{-\pi s}\frac{1}{s^2 + 2s + 2} + \frac{s + 2}{s^2 + 2s + 2}$$

We do each term separately:

$$\text{Let } H(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s + 1)^2 + 1} \quad \text{then} \quad h(t) = \mathrm{e}^{-t}\sin(t)$$

The inverse Laplace transform of the first term is then $u_\pi(t)h(t - \pi)$.

Similarly,
$$\frac{s + 2}{s^2 + 2s + 2} = \frac{s + 1 + 1}{(s + 1)^2 + 1} = \frac{s + 1}{(s + 1)^2 + 1} + \frac{1}{(s + 1)^2 + 1}$$

so the inverse Laplace here is $\mathrm{e}^{-t}(\cos(t) + \sin(t))$.

The overall solution: $y(t) = \mathrm{e}^{-t}(\cos(t) + \sin(t)) + u_\pi(t)\left(\mathrm{e}^{-(t-\pi)}\sin(t - \pi)\right)$

2. Problem 2: $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi)$, zero ICs.

$$(s^2 + 4)Y = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right) \quad \Rightarrow \quad Y = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)\frac{1}{s^2 + 4}$$

Notice that, if $H(s) = \frac{1}{s^2+4}$, and we find $h(t)$, then the solution is $u_\pi(t)h(t - \pi) - u_{2\pi}(t)h(t - 2\pi)$.

Solving for $h(t)$:
$$H(s) = \frac{1}{s^2 + 4} = \frac{1}{2}\frac{2}{s^2 + 4} \qquad h(t) = \frac{1}{2}\sin(2t)$$

Just for practice, you might write the overall solution,

$$y(t) = \frac{1}{2}\left(u_\pi(t)\sin(2(t - \pi)) - u_{2\pi}(t)\sin(2(t - 2\pi))\right)$$

as a piecewise defined function. Notice that $\sin(2t)$ has a period of π.

3. Problem 3: $y'' + 3y' + 2y = \delta(t - 5) + u_{10}(t)$ with ICs $y(0) = 0, y'(0) = \frac{1}{2}$.

$$s^2Y - s0 - \frac{1}{2} + 3(sY - 0) + 2Y = \mathrm{e}^{-5s} + \frac{\mathrm{e}^{-10s}}{s} \quad \Rightarrow \quad Y = \frac{\mathrm{e}^{-5s}}{s^2 + 3s + 2} + \frac{\mathrm{e}^{-10s}}{s(s^2 + 3s + 2)} + \frac{1/2}{s^2 + 3s + 2}$$

The two functions we need to invert:

$$H_1(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s + 2)(s + 1)} = \frac{1}{s + 2} - \frac{1}{s + 1} \quad \Rightarrow \quad h_1(t) = \mathrm{e}^{-2t} - \mathrm{e}^{-t}$$

and

$$H_2(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+2)(s+1)} = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s+2} - \frac{1}{s+1} \quad \Rightarrow \quad h_2(t) = \frac{1}{2} + \frac{1}{2}\mathrm{e}^{-2t} - \mathrm{e}^{-t}$$

With this notation, its easy to write the solution:

$$y(t) = \frac{1}{2}h_1(t) + u_5(t)h_1(t-5) + u_{10}(t)h_2(t-10)$$

4. Problem 4: $y'' - y = -20\delta(t-3)$, $y(0) = 1, y'(0) = 0$.

$$s^2Y - s - Y = -20\mathrm{e}^{-3s} \quad \Rightarrow \quad Y = -20\mathrm{e}^{-3s}\frac{1}{s^2-1} + \frac{s}{s^2-1}$$

   Note that we could factor $s^2 - 1$ as $(s+1)(s-1)$ and perform partial fractions, but in this case we can use Table Entries 7, 8 directly:

$$y(t) = \cosh(t) - 20u_3(t)\sinh(t-3)$$

5. Problem 5: $y'' + 2y' + 3y = \sin(t) + \delta(t - 3\pi)$, $y(0) = 0, y'(0) = 0$.

$$(s^2+2s+3)Y = \frac{1}{s^2+1} + \mathrm{e}^{-3\pi s} \quad \Rightarrow \quad Y = \frac{1}{(s^2+1)(s^2+2s+3)} + \mathrm{e}^{-3s}\frac{1}{s^2+2s+3}$$

   We'll take the partial fractions on first:

$$H_1(s) = \frac{1}{(s^2+1)(s^2+2s+3)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+3} = -\frac{1}{4}\frac{s-1}{s^2+1} + \frac{1}{2}\frac{s+1}{s^2+2s+3} =$$

$$-\frac{1}{4}\left(\frac{s}{s^2+1} - \frac{1}{s^2+1}\right) + \frac{1}{2}\frac{s+1}{(s+1)^2+2} \quad \Rightarrow \quad h_1(t) = -\frac{1}{4}\left(\cos(t) - \sin(t)\right) + \frac{1}{2}\mathrm{e}^{-t}\cos(\sqrt{2}t)$$

   For the second term, let

$$H_2(t) = \frac{1}{s^2+2s+3} = \frac{1}{(s+1)^2+2} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2} \quad \Rightarrow \quad h_2(t) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$$

   Overall, the solution is then:

$$y(t) = h_1(t) + u_{3\pi}(t)h_2(t - 3\pi)$$

   As a double-check, you might notice that before time 3π, $-\frac{1}{4}\left(\cos(t) - \sin(t)\right)$ is the particular part of the solution, and $\frac{1}{2}\mathrm{e}^{-t}\cos(\sqrt{2}t)$ is the homogeneous part of the solution.

6. Problems 6, 8 are very similar to 1-5.

7. For problems 17-21, notice that striking the system will "activate" the homogeneous solution, which is otherwise 0. In this case, the homogeneous solution is $C_1\sin(t) + C_2\cos(t)$, which is periodic with period 2π.

8. Problem 17. Recall that the Laplace operator is linear. Therefore,

$$\mathcal{L}\left(\sum_{k=1}^{20}\delta(t-k\pi\right) = \sum_{k=1}^{20}\mathcal{L}(\delta(t-k\pi)) = \sum_{k=1}^{20}\mathrm{e}^{-k\pi s}$$

Put it together with $\mathcal{L}(y''+y)$:

$$(s^2+1)Y = \sum_{k=1}^{20}\mathrm{e}^{-k\pi s} \quad \Rightarrow \quad Y = \sum_{k=1}^{20}\left(\mathrm{e}^{-k\pi s}\frac{1}{s^2+1}\right)$$

The inverse Laplace transform is:

$$y(t) = \sum_{k-=1}^{20} u_{k\pi}(t)h(t-k\pi) = \sum_{k=1}^{20} u_{k\pi}(t)\sin(t-k\pi)$$

We might write out the first few terms of this to see what it might look like:

$$y(t) = u_\pi(t)\sin(t-\pi)+u_{2\pi}(t)\sin(t-2\pi)+u_{3\pi}(t)\sin(t-3\pi)+u_{4\pi}(t)\sin(t-4\pi)+\cdots u_{20\pi}\sin(t-20\pi)$$

In piecewise form,

$$y(t) = \begin{cases} 0 & \text{if } t<\pi \\ -\sin(t) & \text{if } \pi<t\le 2\pi \\ 0 & \text{if } 2\pi<t\le 3\pi \\ -\sin(t) & \text{if } 3\pi<t\le 4\pi \\ 0 & \text{if } 4\pi<t\le 5\pi \\ \vdots & \vdots \\ -\sin(t) & \text{if } 19\pi<t\le 20\pi \\ 0 & \text{if } t>20\pi \end{cases}$$

IMPORTANT NOTE: The new version of Maple gave an incorrect solution (that was posted online). The correct solution is given, and its plot (from an old version of Maple!) is given in Figure 1.

9. Problem 19. The forcing function is the only thing that changed from Problem 17:

$$(s^2+1)Y = \sum_{k=1}^{20}\mathrm{e}^{-\frac{k\pi}{2}} \quad \Rightarrow \quad Y = \sum_{k=1}^{20}\left(\mathrm{e}^{-\frac{k\pi}{2}}\cdot\frac{1}{s^2+1}\right)$$

Inverting the transform,

$$y(t) = \sum_{k=1}^{20} u_{k\pi/2}(t)\sin(t-k\pi/2)$$

Before writing this in piecewise form, you should verify (by graphing) that:

$$\begin{array}{l|l} \sin(t-\pi/2) = -\cos(t) & \sin(t-3\pi/2) = \cos(t) \\ \sin(t-\pi) = -\sin(t) & \sin(t-2\pi) = \sin(t) \end{array}$$

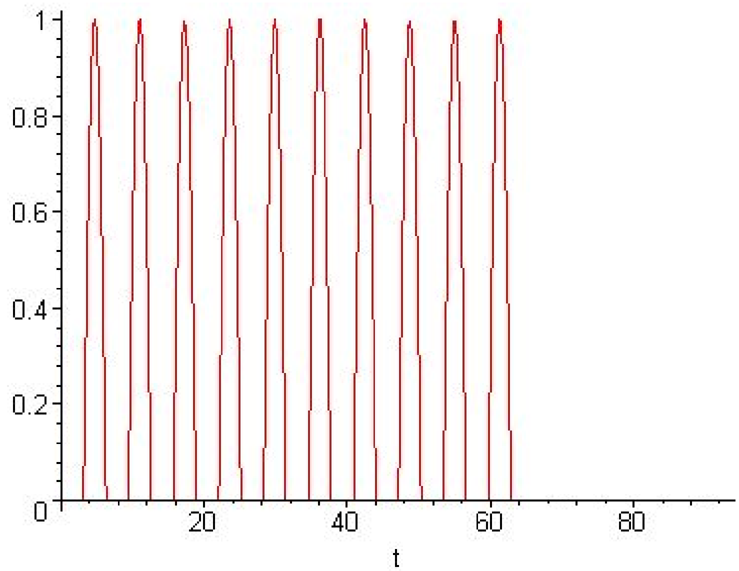

Figure 1: Correct plot of the solution for Exercise 17. The new version of Maple (as posted online) gave an incorrect solution.

Therefore,

$$y(t) = \begin{cases} 0 & \text{if } t < \pi/2 \\ -\cos(t) & \text{if } \pi/2 < t \leq \pi \\ -\cos(t) - \sin(t) & \text{if } \pi/2 < t \leq \pi \\ -\sin(t) & \text{if } \pi < t \leq 3\pi/2 \\ 0 & \text{if } 3\pi/2 < t \leq 2\pi \\ \vdots & \vdots \\ -\sin(t) & \text{if } 19\pi/2 < t \leq 10\pi \\ 0 & \text{if } t > 10\pi \end{cases}$$

10. Problem 21. Finally, we probably can guess what will happen next:

$$y(t) = \sum_{k=1}^{15} u_{(2k-1)\pi} \sin(t - (2k-1)\pi)$$

We notice that the following sine functions are appearing in the sum:

$$\sin(t-\pi), \quad \sin(t-3\pi), \quad \sin(t-5\pi), \quad \sin(t-7\pi), \ldots$$

These are all equivalent to $-\sin(t)$ (verify graphically). Writing y in piecewise form:

$$y(t) = \begin{cases} 0 & \text{if } t < \pi \\ -\sin(t) & \text{if } \pi < t \leq 3\pi \\ -2\sin(t) & \text{if } 3\pi < t \leq 5\pi \\ -3\sin(t) & \text{if } 5\pi < t \leq 7\pi \\ \vdots & \vdots \\ -14\sin(t) & \text{if } 27\pi/2 < t \leq 29\pi \\ -15\sin(t) & \text{if } t > 29\pi \end{cases}$$