Solutions, Section 9.3

For the graphs, see the Maple Worksheet on our class website. To distinguish derivatives, we'll use a dot to denote time derivatives, $\dot{x} = dx/dt$, for example.

For problems 5, 6, 7 and 9, the eigenvalues are optional (we can get all the necessary info from the Poincaré diagram).

1. Problem 5. For the equilibria, recall that if $x = y$ and $x = -y$, then $x = y = 0$.

$$\begin{array}{ll} \dot{x} & = (2+x)(y-x) \\ \dot{y} & = (4-x)(y+x) \end{array} \Rightarrow \text{ Equilibria } \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

The Jacobian is $\begin{bmatrix} y-2x-2 & 2+x \\ -y-2x+4 & 4-x \end{bmatrix}$ Evaluate at the equilibria and classify:

- At the origin,

$$\begin{bmatrix} -2 & 2 \\ 4 & 4 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & 2 \\ \text{Det:} & -16 \\ \text{Disc:} & 68 \end{array} \Rightarrow \text{ Saddle}$$

- At the point $(-2, 2)^T$,

$$\begin{bmatrix} 4 & 0 \\ 6 & 6 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & 10 \\ \text{Det:} & 24 \\ \text{Disc:} & 4 \end{array} \Rightarrow \text{ Source}$$

- At the point $(4, 4)^T$,

$$\begin{bmatrix} -6 & 6 \\ -8 & 0 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & -6 \\ \text{Det:} & 48 \\ \text{Disc:} & -156 \end{array} \Rightarrow \text{ Spiral Sink}$$

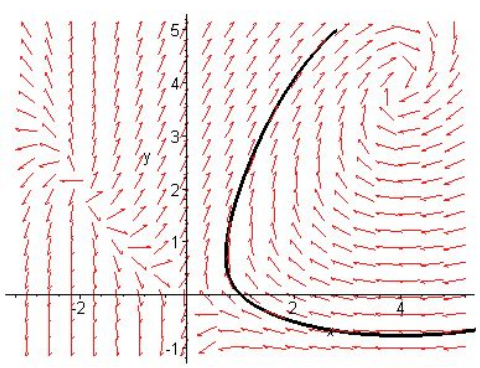

Figure 1: Direction field for Problem 5. We see the saddle at the origin, the spiral sink at $(4, 4)^T$ and the source at $(-2, 2)^T$.

2. Problem 6.

$$\begin{aligned} \dot{x} &= x - x^2 - xy \\ \dot{y} &= 3y - xy - 2y^2 \end{aligned} \Rightarrow \text{ Equilibria } \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3/2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The Jacobian is $\begin{bmatrix} 1-2x-y & -x \\ -y & 3-x-4y \end{bmatrix}$ Evaluate at the equilibria and classify:

- At the origin,

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & 4 \\ \text{Det:} & 3 \\ \text{Disc:} & 4 \end{array} \Rightarrow \text{Source}$$

- At the point $(1,0)^T$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & 1 \\ \text{Det:} & -2 \\ \text{Disc:} & 9 \end{array} \Rightarrow \text{Saddle}$$

- At the point $(0,3/2)^T$,

$$\begin{bmatrix} -1/2 & 0 \\ -3/2 & -3 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & -7/2 \\ \text{Det:} & 3/2 \\ \text{Disc:} & 25/4 \end{array} \Rightarrow \text{Sink}$$

- At the point $(-1,2)^T$,

$$\begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & -3 \\ \text{Det:} & -2 \\ \text{Disc:} & 17 \end{array} \Rightarrow \text{Saddle}$$

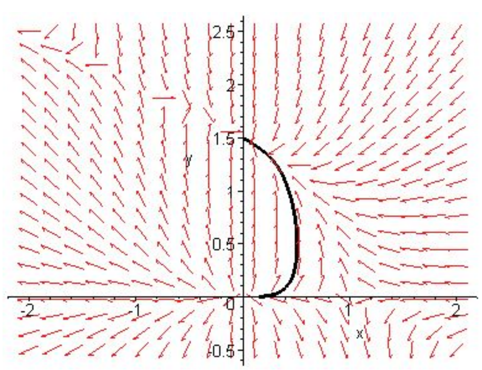

Figure 2: Direction field for Problem 6. We see the source at the origin, the saddles at $(1,0)^T$ and $(-1,2)^T$, and the sink at $(0,3/2)^T$.

3. Problem 7:

$$\begin{aligned} \dot{x} &= 1 - y \\ \dot{y} &= x^2 - y^2 \end{aligned} \Rightarrow \text{Equilibria } \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The Jacobian is $\begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix}$ Evaluate at the equilibria and classify:

- At the point $(-1,1)^T$,

$$\begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & -2 \\ \text{Det:} & -2 \\ \text{Disc:} & 12 \end{array} \Rightarrow \text{Saddle}$$

- At the point $(1,1)^T$,

$$\begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & -2 \\ \text{Det:} & 2 \\ \text{Disc:} & -4 \end{array} \Rightarrow \text{Spiral Sink}$$

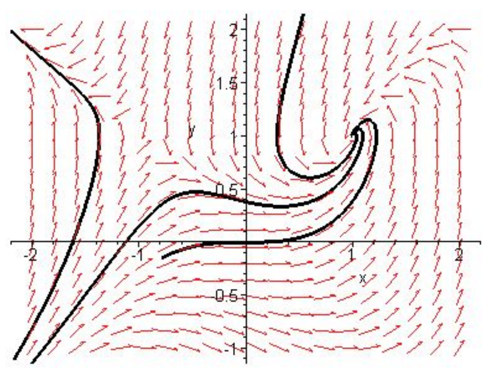

Figure 3: Direction field for Problem 7. We see the saddle at $(-1,1)^T$ and the spiral sink at $(1,1)^T$.

4. Problem 9.

$$\begin{aligned} \dot{x} &= -(x-y)(1-x-y) \\ \dot{y} &= x(2+y) \end{aligned} \Rightarrow \text{Equilibria } \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

The Jacobian is $\begin{bmatrix} -1+2x & 1-2y \\ 2+y & x \end{bmatrix}$ Evaluate at the equilibria and classify:

- At the origin,

$$\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & -1 \\ \text{Det:} & -2 \\ \text{Disc:} & 9 \end{array} \Rightarrow \text{Saddle}$$

- At the point $(0,1)^T$,

$$\begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & -1 \\ \text{Det:} & 3 \\ \text{Disc:} & -11 \end{array} \Rightarrow \text{Spiral Sink}$$

- At the point $(-2,-2)^T$,

$$\begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & -7 \\ \text{Det:} & 10 \\ \text{Disc:} & 9 \end{array} \Rightarrow \text{Sink}$$

- At the point $(3,-2)^T$,

$$\begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & 8 \\ \text{Det:} & 15 \\ \text{Disc:} & 4 \end{array} \Rightarrow \text{Source}$$

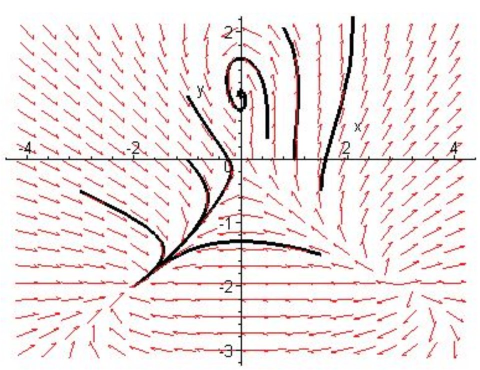

Figure 4: Direction field for Problem 9. We see the saddle at the origin, the spiral sink at $(0,1)^T$, the (regular) sink at $(-2,-2)$ and the source at $(3,-2)^T$.

5. Problem 18:

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -2y + x^3 \end{aligned} \Rightarrow \text{Equilibria} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The Jacobian is $\begin{bmatrix} 1 & 0 \\ 3x^2 + y & -2 \end{bmatrix}$ Evaluate at the equilibria and classify:

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Trace:} & -1 \\ \text{Det:} & -2 \\ \text{Disc:} & 9 \end{array} \Rightarrow \text{Saddle}$$

In this case, we are able to compute solutions:

$$\frac{dy}{dx} = \frac{-2y + x^3}{x} \quad \Rightarrow \quad y' + \frac{2}{x} y = x^2$$

Use the method of integrating factors, $\mathrm{e}^{\int p(x)\,dx} = x^2$, and

$$(x^2 y)' = x^4 \quad \Rightarrow \quad y = \frac{1}{5}x^3 + \frac{C}{x^2}$$

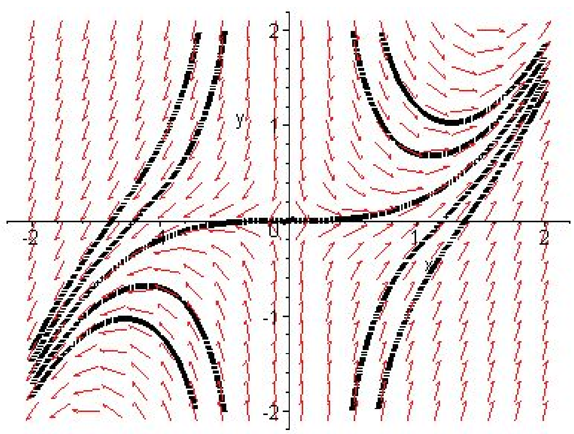

Figure 5: Direction field for Problem 18, and we also see solution curves. The heavy black curves are the contours for the function $x^2 y - \frac{1}{5}x^5$ for contours $-1, -1/2, 0, 1/2, 1$. Notice that, at the contour 0, we simply have the curve $y = x^3$. The linearization predicted a saddle, which we also see (and this is a good example of how the nonlinear system "tweaks" the linear system).